WTF?

Min-cost matching, restated algebraically.

Variable \( x_{uv} \)  

Intended meaning \( x_{uv} = \begin{cases} 1 & \text{if } (u,v) \in M \\ 0 & \text{otherwise} \end{cases} \)

\[
\begin{align*}
\min & \quad \sum_{u \in L, v \in R} c(uv) \cdot x_{uv} \\
\text{s.t.} & \quad \sum_v x_{uv} = 1 \quad \forall u \in L \\
& \quad \sum_u x_{uv} = 1 \quad \forall v \in R \\
& \quad x_{uv} \in \{0,1\} \\
\end{align*}
\]

If \( x_{uv} \notin \{0,1\} \), then e.g.

\[ \begin{array}{c}
\begin{array}{cccc}
\text{min} & \sum_{u \in L, v \in R} c(uv) \cdot x_{uv} \\
\text{s.t.} & \sum_v x_{uv} = 1 & \forall u \in L \\
& \sum_u x_{uv} = 1 & \forall v \in R \\
& x_{uv} \geq 0 \\
\end{array}
\end{array} \]

A solution of these constraints is called a fractional perfect matching.
Birkhoff–von Neumann Thm. Every "fractional perfect matching" in a bipartite graph is a convex combination of perfect matchings.

How would one show that a specified perfect matching has minimum cost among all fractional perfect matchings?

E.g.,

\[ y_u = 2 \quad x_{v_{11}} + x_{v_{12}} = 2 \]
\[ y_u = -1 \quad -x_{v_{11}} - x_{v_{12}} = -1 \]
\[ y_{v'} = 4 \quad 4x_{u_{v_{11}}} + 4x_{u_{v_{12}}} = 4 \]
\[ y_{v'} = 4 \]

Exercise: For the minimum k edge matching problem, complete this story by expressing it algebraically and explaining where properties (3)–(4) come from.