Lecture 16 Max Flow

Suppose we are given a tuple $G = (V, c, s, t)$, where V is a set of *vertices*, $s, t \in V$ are distinguished vertices called the *source* and *sink* respectively, and c is a function $c : V^2 \to \mathcal{R}_+$ assigning a nonnegative real *capacity* to each pair of vertices. We make G into a directed graph by defining the set of directed edges

$$E \quad = \quad \{(u, v) \mid c(u, v) > 0\} \ .$$

Intuitively, we can think of the edges as wires or pipes along which electric current or fluid can flow; the capacity $c(e)$ represents the carrying capacity of the wire or pipe, say in amps or gallons per minute. The *max flow problem* is to determine the maximum possible flow that can be pushed from s to t, and to find a routing that achieves this maximum. The following definition is intended to capture the intuitive idea of a *flow*.

Definition 16.1 A function $f : V^2 \to R$ is called a *flow* if the following three conditions are satisfied:

(a) *skew symmetry*: for all $u, v \in V$,

$$f(u, v) \quad = \quad -f(v, u) \ ;$$

(b) *conservation of flow at interior vertices*: for all vertices u not in $\{s, t\}$,

$$\sum_{v \in V} f(u, v) \quad = \quad 0 \ ;$$

(c) *capacity constraints*: $f \leq c$ pointwise; *i.e.*, for all u, v,

$$f(u,v) \leq c(u,v) .$$

We say that (u, v) is *saturated* if $f(u, v) = c(u, v)$. □

If we think of edges (u, v) for which $f(u, v) > 0$ as carrying flow *out of* u, and edges (u, v) for which $f(u, v) < 0$ (or equivalently by (a), $f(v, u) > 0$) as carrying flow *into* u, then condition (b) says that the total flow out of any interior vertex is equal to the total flow into that vertex, or in other words, the *net flow* (total flow out minus total flow in) at any interior vertex is 0.

It follows from (a) that $f(u, u) = 0$ for any vertex u.

Figure 1 illustrates a graph with capacities c (ordinary typeface) and a flow f on that graph (italic). Edges not shown have a capacity of 0 and a flow that is the negative of the flow in the opposite direction; *e.g.*, $c(u, s) = 0$ and $f(u, s) = -4$. If neither an edge nor its opposite is shown (*e.g.* (s, t)), then the capacities and flows in both directions are 0.

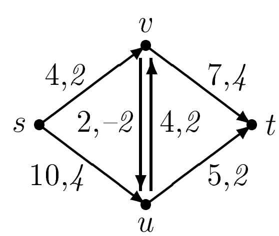

Figure 1

Definition 16.2 An *s, t-cut* (or just *cut*, when s, t are understood) is a pair A, B of disjoint subsets of V whose union is V such that $s \in A$, $t \in B$. The *capacity* of the cut A, B, denoted $c(A, B)$, is

$$c(A,B) = \sum_{u \in A,\ v \in B} c(u,v) ,$$

i.e., the total capacity of the edges from A to B. If f is a flow, we define the *flow across the cut* A, B to be

$$f(A,B) = \sum_{u \in A,\ v \in B} f(u,v) .$$

□

Note that by condition (a) of Definition 16.1, $f(A, B)$ gives the *net flow* across the cut from A to B; that is, the sum of the positive flow values on edges from A to B minus the sum of the positive flow values on edges from B to A.

Definition 16.3 The *value* of a flow f, denoted $|f|$, is defined to be

$$\begin{aligned} |f| &= f(\{s\}, V - \{s\}) \\ &= \sum_{v \in V} f(s,v) \ , \end{aligned}$$

or in other words the net flow out of s. □

In the example of Figure 1, $|f| = 6$.

Although Definition 16.3 defines the value of the flow f with respect to the cut $\{s\}, V - \{s\}$, the flow value will be the same no matter where it is measured:

Lemma 16.4 *For any s,t-cut A, B and flow f,*

$$|f| = f(A, B) \ .$$

Proof. Induction on the cardinality of A, using condition (b) of Definition 16.1. □

In particular,

$$f(\{s\}, V - \{s\}) = f(V - \{t\}, \{t\}) \ ,$$

which says that the net flow out of s equals the net flow into t.

The flow across any cut surely cannot exceed the capacity of the cut. This is expressed in the following lemma:

Lemma 16.5 *For any s,t-cut A, B and flow f,*

$$|f| \leq c(A, B) \ .$$

Proof. Lemma 16.4 and condition (c). □

The main result of this lecture will be the *Max Flow-Min Cut Theorem*, which states that the minimum cut capacity is achieved by some flow; *i.e.*, the inequality in Lemma 16.5 is an equality for some cut A, B and some flow f^*. The flow f^* necessarily has maximum value among all flows on G by Lemma 16.5, and is called a *max flow*. The flow f^* is not unique, but its value is.

16.1 Residual Capacity

Definition 16.6 Given a flow f on G with capacities c, we define the *residual capacity function* $r : V^2 \to R$ to be the pointwise difference

$$r = c - f \ .$$

The *residual graph* associated with $G = (V, E, c)$ and flow f is the graph $G_f = (V, E_f, r)$, where

$$E_f = \{(u,v) \mid r(u,v) > 0\} \ .$$

□

The residual capacity $r(u,v)$ represents the amount of additional flow that could be pushed along the edge (u,v) without violating the capacity constraint (c) of Definition 16.1. In case the flow $f(u,v)$ is negative, this "additional flow" could involve backing off the positive flow from v to u. For example, if $c(u,v) = 8$ and $f(u,v) = 6$, and $(v,u) \notin G$ so that $c(v,u) = 0$, then $r(u,v) = 2$ and $r(v,u) = c(v,u) - f(v,u) = 0 - (-6) = 6$. The residual graph for the flow in Figure 1 is given in Figure 2 below.

Note that the residual graph G_f can have an edge where there was none in G. However, G_f has no edges (u,v) where neither (u,v) nor (v,u) were present in G, so $|E_f| \le 2 \cdot |E|$.

Intuitively, the formation of the residual graph translates the problem by making f the new origin (zero flow). Solving the residual flow problem is tantamount to solving the original flow problem; a solution to the residual flow problem can be added to f to obtain a solution to the original problem. This observation is formalized in the following lemma.

Lemma 16.7 *Let f be a flow in G, and let G_f be its residual graph.*

(a) The function f' is a flow in G_f iff $f + f'$ is a flow in G.

(b) The function f' is a max flow in G_f iff $f + f'$ is a max flow in G.

(c) The value function is additive; i.e., $|f + f'| = |f| + |f'|$ and $|f - f'| = |f| - |f'|$.

(d) If f is any flow and f^* a max flow in G, then the value of a max flow in G_f is $|f^*| - |f|$.

Proof.

(a) Since f is a flow, it satisfies skew symmetry ($f(u,v) = -f(v,u)$) and conservation at interior vertices ($\sum_v f(u,v) = 0$). Thus f' satisfies these properties iff $f + f'$ does. To show that the capacity constraints are satisfied, recall that the capacities of G_f are given by $r = c - f$, where c is the capacity function of G. Then

$$\begin{aligned} f' \le r \quad &\text{iff} \quad f' \le c - f \\ &\text{iff} \quad f + f' \le c\,. \end{aligned}$$

(b) This follows directly from (a).

(c) By the definition of flow value,

$$\begin{aligned} |f \pm f'| &= \sum_v (f(s,v) \pm f'(s,v)) \\ &= \sum_v f(s,v) \pm \sum_v f'(s,v) \\ &= |f| \pm |f'|\,. \end{aligned}$$

(d) This follows directly from (b) and (c).

□

16.2 Augmenting Paths

Definition 16.8 Given G and flow f on G, An *augmenting path* is a directed path from s to t in the residual graph G_f. □

An augmenting path represents a sequence of edges on which the capacity exceeds the flow, *i.e.*, on which the flow can be increased. As observed above, on some edges this "increase" may actually involve decreasing a positive flow in the opposite direction.

Figure 2 illustrates the residual graph associated with the flow in the example of Figure 1 and an augmenting path. The minimum capacity of any edge in this path is 2, so the flow can be increased on these edges by 2, resulting in a new flow in the original graph with value 2 greater than that of $|f|$. Note that the "increase" on (u, v) is essentially a decrease of a positive flow on (v, u).

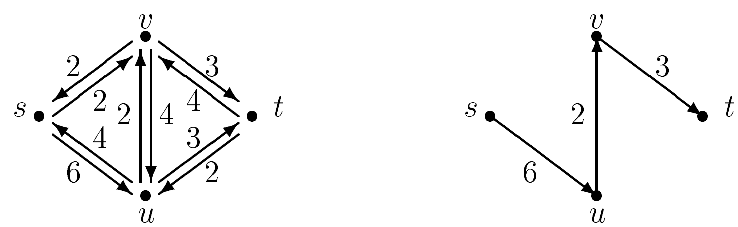

Figure 2

We are now ready to state and prove the main theorem of this lecture:

Theorem 16.9 (Max Flow-Min Cut Theorem [34]) *The following three statements are equivalent:*

(a) f is a max flow in $G = (V, E, c)$;

(b) there is an s,t-cut A, B with $c(A, B) = |f|$;

(c) there does not exist an augmenting path.

Proof.

(b) → (a) This is immediate from Lemma 16.5.

(a) → (c) Suppose there is an augmenting path $u_0, u_1, \ldots, u_n$ with $s = u_0$ and $t = u_n$. Let

$$d = \min\{r(u_i, u_{i+1}) \mid 0 \le i < n\} > 0 .$$

The quantity d is the smallest residual capacity along the augmenting path and is called the *bottleneck capacity*. An edge along the augmenting path with that capacity is called a *bottleneck edge*. Define the following flow g in the residual graph G_f:

$$\begin{aligned} g(u_i, u_{i+1}) &= d, \quad 0 \le i < n \\ g(u_{i+1}, u_i) &= -d, \quad 0 \le i < n \\ g(u, v) &= 0, \quad \text{for all other pairs } (u, v). \end{aligned}$$

Then g is a flow in G_f with value d. By Lemma 16.7, $f+g$ is a flow in G and $|f+g| = |f| + |g| = |f| + d$.

(c) $\rightarrow$ (b) Assume there is no augmenting path. Let A consist of all vertices reachable from s by paths in the residual graph. Let $B = V - A$. There are no edges in the residual graph from A to B; thus in G, all edges from A to B are saturated, *i.e.* $f(u,v) = c(u,v)$. It follows from Lemma 16.4 that $c(A,B) = |f|$. $\square$