Theorem  The dual matroid is a matroid.

Proof. Let $M = (S, I)$ be matroid. A basis of $M$ is a maximal independent set in $I$. The dual of $M$ is $M^* = (S, I^*)$, where

$$I^* = \{ A \subseteq S \mid I \cap A = \emptyset \text{ for some basis } I \text{ of } M \}.$$  

Note that $M^{**} = M$ and that the bases of $M^*$ are the complements of the bases of $M$. We show that $M^*$ is a matroid. It is clearly closed downward under $\subseteq$, so we need only show

$$\forall A, B \in I^* \mid |A| < |B| \Rightarrow \exists x \in B - A \ A \cup \{x\} \in I^*.$$  

Suppose $A, B \in I^*$ and $|A| < |B|$. There exist bases $I, J$ of $M$ such that $A \cap J = \emptyset$ and $B \cap I = \emptyset$. Complete $I - A$ to a basis $K$ of $M$ by adding elements of $J$. Then $A \cap K = \emptyset$ and

$$|K| = |K - I| + |K \cap I| = |K - I| + |I - A| = |K - I| + |I| - |I \cap A|,$$

and $|K| = |I|$, so $|K - I| = |I \cap A|$ and

$$|K \cap (B - A)| \leq |K - I| = |I \cap A| \leq |A - B| < |B - A|.$$  

There must exist $x \in (B - A) - K$, so $(A \cup \{x\}) \cap K = \emptyset$, which says that $A \cup \{x\} \in I^*$.

\[\qed\]

Theorem  Cuts in $M$ are cycles in $M^*$ and vice versa.

Proof. By duality, we only need to show one of the two statements.

$$A \text{ is a cut of } M \iff A \text{ is a minimal set intersecting all bases } I \text{ of } M$$  

$$\iff A \text{ is a minimal set intersecting all } S - J \text{ for bases } J \text{ of } M^*$$  

$$\iff A \text{ is a minimal set not contained in any basis } J \text{ of } M^*$$  

$$\iff A \text{ is a minimal dependent set in } M^*$$  

$$\iff A \text{ is a cycle of } M^*.$$  

\[\qed\]

Corollary  The blue (respectively, red) rule of $M$ is the red (respectively, blue) rule of $M^*$ with the order of the weights reversed.