4.2 Directed DFS

The DFS procedure on directed graphs is similar to DFS on undirected graphs, except that we only follow edges from sources to sinks. Four types of edges can result:

- *tree edges* to a vertex not yet visited
- *back edges* to an ancestor
- *forward edges* to a descendant previously visited
- *cross edges* to a vertex previously visited that is neither an ancestor nor a descendant.

There can be no cross edges to a higher numbered vertex; such an edge would have been a tree edge. If we mark the vertex y when the tree edge (x, y) is popped to indicate that the subtree below y has been completely explored, we can recognize each of these four cases when we explore the edge (u, v) by checking marks and comparing DFS numbers:

(u, v) *is a*	*if*
tree edge	DFS(v) does not exist
back edge	DFS(v) $<$ DFS(u) and v is not marked
forward edge	DFS(v) $>$ DFS(u)
cross edge	DFS(v) $<$ DFS(u) and v is marked

The directed DFS tree can be constructed in linear time; see [3, 78] for details.

The first application of directed DFS is determining acyclicity:

Theorem 4.8 *A directed graph is acyclic iff its DFS forest has no back edges.*

Proof. If there is a back edge, the graph is surely cyclic. Conversely, if there are no back edges, consider the *postorder* numbering of the DFS forest: traverse the forest in depth-first order, but number the vertices in the order they are *last* seen. Then tree edges, forward edges, and cross edges all go from higher numbered to lower numbered vertices, so there can be no cycles. □

4.3 Strong Components

Definition 4.9 Let $G = (V, E)$ be a directed graph. For $u, v \in V$, define $u \equiv v$ if u and v lie on a directed cycle in G. This is an equivalence relation, and its equivalence classes are called *strongly connected components* or just *strong components*. A graph G is said to be *strongly connected* if for any pair of vertices u, v there is a directed cycle in G containing u and v; *i.e.*, if G has only one strong component. □

The strong components of a directed graph can be computed in linear time using directed depth-first search. The algorithm is similar to the algorithm for biconnected components in undirected graphs; see [3] for details.

4.4 Strong Components and Partial Orders

Strong components are important in the representation of partial orders. Finite partial orders are often represented as the reflexive transitive closures E^* of dags $G = (V, E)$ (recall $(u, v) \in E^*$ iff there exists an E-path from u to v of length 0 or greater). If G is not acyclic, then the relation E^* does not satisfy the antisymmetry law, and is thus not a partial order. However, it is still reflexive and transitive. Such a relation is called a *preorder* or sometimes a *quasiorder*.

Given an arbitrary preorder $(P, \preceq)$, define $x \approx y$ if $x \preceq y$ and $y \preceq x$. This is an equivalence relation, and we can collapse its equivalence classes into single points to get a partial order. This construction is called a *quotient construction*. Formally, let $[x]$ denote the $\approx$-class of x and let $P/\!\approx$ denote the set of all such classes; *i.e.*,

$$\begin{aligned} [x] &= \{y \mid y \approx x\} \\ P/\!\approx &= \{[x] \mid x \in P\} \, . \end{aligned}$$

The preorder $\preceq$ induces a preorder, also denoted $\preceq$, on $P/\approx$ in a natural way: $[x] \preceq [y]$ if $x \preceq y$ in P. (The choice of x and y in their respective equivalence classes doesn't matter.) It is easily shown that the preorder $\preceq$ is actually a partial order on $P/\approx$; intuitively, by collapsing equivalence classes, we identified those elements that caused antisymmetry to fail.

Forming the strong components of a directed (not necessarily acyclic) graph $G = (V, E)$ allows us to perform this operation effectively on the preorder (V, E^*). We form a quotient graph $G/\equiv$ by collapsing the strong components of G into single vertices:

$$\begin{aligned} [v] &= \{u \mid u \equiv v\} \text{ (the strong component of } v) \\ V/\equiv &= \{[v] \mid v \in V\} \\ E' &= \{([u],[v]) \mid (u,v) \in E\} \\ G/\equiv &= (V/\equiv, E') . \end{aligned}$$

It is not hard to show that $G/\equiv$ is acyclic. Moreover,

Theorem 4.10 *The partial orders $(V/\approx, E^*)$ and $(V/\equiv, (E')^*)$ are isomorphic.*

In other words, the partial order represented by the collapsed graph is the same as the collapse of the preorder represented by the original graph.