V. Vazirani, *Approximation algorithms*, Springer, 2003.

9 Bin Packing

Consider the following problem.

Problem 9.1 (Bin packing) Given n items with sizes $a_1, \ldots, a_n \in (0, 1]$, find a packing in unit-sized bins that minimizes the number of bins used.

This problem finds many industrial applications. For instance, in the stock-cutting problem, bins correspond to a standard length of paper and items correspond to specified lengths that need to be cut.

It is easy to obtain a factor 2 approximation algorithm for this problem. For instance, let us consider the algorithm called First-Fit. This algorithm considers items in an arbitrary order. In the ith step, it has a list of partially packed bins, say $B_1, \ldots, B_k$. It attempts to put the next item, a_i, in one of these bins, in this order. If a_i does not fit into any of these bins, it opens a new bin B_{k+1}, and puts a_i in it. If the algorithm uses m bins, then at least $m-1$ bins are more than half full. Therefore,

$$\sum_{i=1}^{n} a_i > \frac{m-1}{2}.$$

Since the sum of the item sizes is a lower bound on OPT, $m - 1 < 2\text{OPT}$, i.e., $m \le 2\text{OPT}$ (see Notes for a better analysis). On the negative side:

Theorem 9.2 *For any $\varepsilon > 0$, there is no approximation algorithm having a guarantee of $3/2 - \varepsilon$ for the bin packing problem, assuming* $\mathbf{P} \neq \mathbf{NP}$.

Proof: If there were such an algorithm, then we show how to solve the **NP**-hard problem of deciding if there is a way to partition n nonnegative numbers $a_1, \ldots, a_n$ into two sets, each adding up to $\frac{1}{2}\sum_i a_i$. Clearly, the answer to this question is 'yes' iff the n items can be packed in 2 bins of size $\frac{1}{2}\sum_i a_i$. If the answer is 'yes' the $3/2 - \varepsilon$ factor algorithm will have to give an optimal packing, and thereby solve the partitioning problem. □

9.1 An asymptotic PTAS

Notice that the argument in Theorem 9.2 uses very special instances: those for which OPT is a small number, such as 2 or 3, even though the number