1 Introduction

Expander graphs are graphs that are both sparse and well connected. By sparse we mean that they have $O(N)$ edges (where $N$ is the number of edges in the graph). There are several different (yet connected) definitions of well connected which we will see throughout the lecture.

2 Vertex Expansion

For the duration of the lecture, we will be considering directed $D$-regular graphs.

Definition 2.1. For a graph $G = (V, E)$, we define the neighbor set of a vertex $u \in V$ as $N(u) = \{v : (u, v) \in E\}$. We define the neighbor set of a set of vertices $S \subseteq V$ as $N(S) = \bigcup_{u \in S} N(u)$.

Definition 2.2. $G$ is a $(K, A)$ vertex expander if for all $S \subseteq V$ such that $|S| \leq K$, $|N(S)| \geq A|S|$.

Notice that the property that $|S| \leq K$ is necessary because otherwise $A = 1$ (because then we can have the case where $S = V$).

Intuitively, a $G$ is a vertex expander if when we look at a subset of vertices $S$, we can always reach more vertices if we take a step from inside the set $S$.

Remark 2.3. Edge expansion is very similar, it just requires you to think about the neighbors as the edges you can reach quickly form $S$ rather than the vertices you can reach quickly from $S$.

Ideally, we would like $D = O(1)$, $A \approx D - 1$ and $K = \Omega(N)$.

We can show the existence of expanders by the probabilistic method.

However, we will cheat slightly and show the existence of bipartite expanders. A bipartite expander only requires that the expansion property only hold for subsets in the left side of the graph. We will also only require that the graph is left $D$-regular.

Theorem 2.4. for all $D = O(1)$, there exists an $\alpha = O(1)$ such that a random left $D$-regular bipartite digraph is an $(\alpha N, D - 1.1)$ vertex expander with high probability.

Let us consider the probability that for a fixed set $S$ with $|S| = k$ that a random bipartite graph violates the expansion property. In other words that $|N(S)| \leq (D - 1.1)K$.

Notice that for $|N(S)| \leq DK - 1.1K$, there must be at least $1.1K$ repetitions in the edges that repeat a node, go to a node which has already been reached by a previous edge.

The probability that a given edge goes to a node that has been covered by a previous edge is bounded above by $\frac{KD}{N}$. Thus, the probability that a given set of edges cover $1.1K$ or more already covered nodes is bounded above by $(\frac{KD}{N})^{1.1K}$. Furthermore, there are $\binom{N}{1.1K}$ choices for the edges which will be repetitions. Thus, for a given $S$, the probability that there will be $1.1K$ or more repetitions is upper bounded by $\binom{N}{1.1K}(\frac{KD}{N})^{1.1K}$.

Now we will look at the probability that there exists any $S$ with $|S| = k$, which violates the expansion property.
\[ \mathbb{P}[\exists S, |S| = K, |N(S)| \leq (D - 1.1)K] \]

By the union bound this is

\[ \leq \left( \frac{N}{K} \right) \left( \frac{N}{1.1K} \right) \left( \frac{KD}{N} \right)^{1.1K} \]

By the approximation for the binomial we can see that the above is less than or equal to

\[ \leq \left( \frac{eN}{K} \right)^K \left( \frac{eKD}{1.1K} \right)^{1.1K} \]

\[ = \left( \frac{e^{2.1}D^{2.2}}{1.1^{1.1}N^{0.1}} \right)^K \]

By making \( \alpha \) very small, we can make \( K \) arbitrarily small, thus we can make the following less than or equal to

\[ \leq 11^{-K} \]

Finally, we can arrive at the probability that the expansion property is not violated for any set of size less than \( K \) as less than or equal to

\[ \sum_{i=1}^{\alpha N} 11^{-1} \leq 0.1 \]

Thus with high probability (at least 90%) our graph is an expander. Thus, it exists.

3 Spectral Expansion

**Definition 3.1.** \( M \) is the random walk matrix of a graph \( G \) if \( M_{i,j} = \frac{\text{number of edges from } i \text{ to } j}{D} \).

\( M_{i,j} \) can be thought of as the probability you go from to node \( j \) if you are at node \( i \) and choose to travel along one of the edges with equal probability.

Let \( \pi = [p_1, p_2, \ldots, p_n] \), where \( p_i \) is the probability that you are at node \( i \).

\( (\pi M) \) is then the probability that you are at node \( i \) after taking a random step in the graph having been in node \( x \) with probability \( p_x \) previously.

Let \( u \) be the vector that represents the uniform distribution on vertices, \( u = \left[ \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N} \right] \)

Then we can define the expansion of the graph as

\[ \lambda(G) = \max_{x \perp u} \frac{\|xM\|}{\|x\|} \]

**Remark 3.2.** Usually \( xM^n \) (the distribution after taking \( n \) steps in the graph) converges to a stationary distribution.

\( \lambda(G) \) can be thought of as how fast you converge to the uniform distribution for any \( \pi \). We will now see why.

\( \pi = u + x \), observe that \( u \perp x \) since \( \|x\| = 0 \)

Therefore \( \pi M = (u + x)M = u + xM \) and we know that \( \|xM^t\| \leq \lambda(G)^t\|x\| \). So how quickly the distribution converges to the uniform distribution is dependent in \( xM^t \) which is bounded by \( \lambda(G) \). So \( \lambda(G) \) tells us how quickly the distribution converges to uniform.
Remark 3.3. If you take an undirected graph and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $M$, then $\lambda(G) = \lambda_2$.

Theorem 3.4. Spectral expansion implies vertex expansion. More precisely, for $\alpha \in [0,1]$, $G$ is a $(\alpha N, \frac{1}{(1-\alpha)\lambda^2 + \alpha})$ vertex expander.

Notice that if $\lambda < 1$, you have $\frac{1}{(1-\alpha)\lambda^2 + \alpha} > 1$, which means we have expansion. Otherwise, we do not.

4 Mixing

Definition 4.1. A graph $G$ has the mixing property if for 2 sets $S$ and $T$ where $|S| = \alpha N$ and $|T| = \beta N$, $\frac{e(S,T)}{ND} \approx \alpha \beta$. $e(S,T)$ is the number of edges between $S$ and $T$.

Notice that in a random graph, you would expect the mixing property to hold.

Theorem 4.2. Spectral Expansion implies Mixing. More precisely $|e(S,T) - \alpha \beta| \leq \lambda p \alpha \beta (1-\alpha)(1-\beta)$ or the more useful bound $|e(S,T) - \alpha \beta| \leq \sqrt{\alpha \beta}$

Note that if $\lambda \approx 0$, then the density between $S$ and $T$ is very close to $\alpha \beta$.

Remark 4.3. Vertex expansion with strong parameters implies spectral expansion. In fact, to a certain degree any of the 3 definitions of expansion given in this lecture imply the other 2.

We will now present the proof that mixing implies spectral expansion

$1_S = [1_1, 1_2, \ldots , 1_n]$, where $1_i = 1$ if and only if $i \in S$.

$e(S,T) = 1_S^T A 1_T$

Where $A$ is the adjacency matrix of $M$. $A = DM$. Therefore

$e(S,T) = 1_S^T DM 1_T$

The above is true because the left hand side is equal to the following

$\sum_{i,j} (1_S)_i (DM)_{i,j} (1_T)_j$

Notice that the expression expression in the sum gives the number of edges between $i$ and $j$ if $i$ is in $S$ and $j$ is in $T$, and zero otherwise. Thus the sum gives the total number of edges between $S$ and $T$.

Recall from linear algebra that we can write any vector $v$ as $kv + v^\perp$, where $k = \sum v_i$.

Using this fact, we can rewrite out expression as

$$(\alpha N v + (1_{\frac{1}{N}}))^T DM (\beta N v + (1_{\frac{1}{N}}))^T$$

Expanding and combing terms we get

$$\alpha \beta N^2 D v^T M v \frac{1}{N} + ((1_S)_T)^T DM (1_T)_T$$

$$= \alpha \beta N^2 D \frac{1}{N} + ((1_S)_T)^T DM (1_T)_T$$
The first term now simplifies to $\alpha \beta N D$. All that remains is to bound the error term $((\mathbb{1}_S)^\perp)^T D M (\mathbb{1}_T)^\perp$.

By Cauchy-Swartz

$$((\mathbb{1}_S)^\perp)^T D M (\mathbb{1}_T)^\perp \leq \|(\mathbb{1}_S)^\perp\| D M \mathbb{1}_T \|

\leq \lambda D \|(\mathbb{1}_S)^\perp\| \mathbb{1}_T \|

By Pythagoras, we know $\|\mathbb{1}_S\| = \sqrt{\alpha(1 - \alpha)N}$. Thus we have

$$\leq \lambda D \sqrt{\alpha(1 - \alpha)N} \sqrt{\beta(1 - \beta)N}

= \lambda N D \sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}$$