8.1 Expanders - Explicit constructions

In this lecture, we formulate ways to take products of graphs to construct larger expander graphs.

New notation: Let a $(N, D, \gamma)$-graph be a $(N, \gamma)$ spectral expander that is $D$-regular.

Our approach: start with small graphs and iteratively construct larger graphs.

8.2 Squaring Graphs

Intuitively, squaring is just 2 hops on the original graph. Note that self loops and multiple edges are allowed in squared graphs.

Formally, if we have a graph $G = (V, E)$, let $G^2 = (V, E')$ be a graph such that, for all $v$ in $V$, the $(i,j)$th neighbor of $v$ is the $j$th neighbor of the $i$th neighbor of $v$, where $i, j \in [D]$ (are numbers from 1 to D).

This operation doesn’t add any nodes, and it squares the number of edges. $A^2$ is the normalized adjacency/random-walk matrix of $G^2$. Hence, $\lambda(G^2) = \lambda(G)^2$.

- degree increases :( 
- nodes remain same :/
- expansion improves :)

8.3 Tensor Products

For $V$ in $R_n$, $W$ in $R_m$, the tensor product of $V$ and $W$ is denoted as $Z = V \otimes W \in R_{n \times m}$. It is a generalization of the outer product.

For two vectors, we define their tensor product to be a matrix, such that $z_{ij} = v_i w_j$, $i \in [n], j \in [m]$. For two matrices $A \in R_{n_1 \times n_2}, B \in R_{m_1 \times m_2}$, the entries of the tensor product $C = A \otimes B$ are as follows:

$C_{i_1i_2j_1j_2} = A_{i_1i_2}B_{j_1j_2}$

Some properties of tensor products:

1. $A \otimes (B + C) = A \otimes B + A \otimes C$
2. in general, $A \otimes B \neq B \otimes A$
3. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ if $AC$ and $BD$ are defined by the standard rules of matrix multiplication
4. $(A \otimes B)^T = A^T \otimes B^T$

With property number 3, if $A_{n \times n}$ and $B_{m \times m}$ are matrices and $C_n$ and $D_m$ are vectors, $A$ effectively acts just on $C$ and $B$ just on $D$. This is a major part of the intuition for why tensor products can be useful.

### 8.4 Tensor Products of Graphs

Suppose we have $G_1, G_2$, such that:
- $G_1$ is an $(N_1, D_1, \gamma_1)$-graph, and its adjacency matrix is $M_1$.
- $G_2$ is an $(N_2, D_2, \gamma_2)$-graph, and its adjacency matrix is $M_2$.

Then, we define the tensor product of $G_1$ and $G_2$ to be $G = G_1 \otimes G_2$.

The adjacency matrix of $G$ is $M_1 \otimes M_2$. The set of vertices of $G$ is $[N_1] \times [N_2]$. $(v, j)$ is a neighbor of $(u, i)$ if $(u, v)$ is in $E_{G_1}$, $(i, j)$ in $E_{G_2}$.

To visualize this, make 4 “clouds” that are copies of the vertices of $G_2$; each cloud represents one vertex of $G_1$. Draw an edge between two vertices in different clouds if the vertices corresponding to the clouds in $G_1$ are connected, and the vertices corresponding to the positions in the intra-cloud graph are connected in $G_2$.

Now, we analyze the spectral expansion of $G$.

The eigenvalues of $A_1 \otimes A_2$ are $\lambda_i(G_1)\lambda_j(G_2), i \in [N_1], j \in [N_2]$ – the largest eigenvalue is $1 \cdot 1$, so the second largest is $1 \cdot \lambda_{G_1}$ or $1 \cdot \lambda_{G_2}$.

$G$ is $(N_1 N_2, D_1 D_2, \min(\gamma_{G_1}, \gamma_{G_2}))$.

- degree increases :
- nodes increase :) 
- expansion remains same :/

There is a more intuitive proof of the spectral expansion for tensor products that helps build the intuition needed to think about the zig-zag product. The rest of this scribed document will be focused on this proof.

#### 8.4.1 Intuitive Proof of Spectral Expansion for Tensor Products

$A = A_1 \otimes A_2$

w.t.s. that $\|Ax\| \leq \lambda \|x\|, x \perp 1_{N_1 N_2}$

$x$ is a long vector, but we’ll think of it as the flattened out form of a matrix that is $N_1 \times N_2$. Think of $x$ as a probability distribution; the $i$th row is the marginal of $x$ on the $i$th cloud.

Write $x$ as $x\parallel + x\perp$, where $x\parallel$ is parallel to $u_{N_2}$ (where $u$ is the normalized all-ones vector) on each cloud. Visualize $x\parallel$ and $x\perp$ as matrices of the same dimension as $x$.

$x\parallel = y \otimes u_{N_2}$, for some unique vector $y$ in $R^{N_1}$. Note that $y$ is perpendicular to $u_{N_1}$.

$Ax\parallel = (A_1 \otimes A_2)(y \otimes u_{N_2}) = (A_1 y \otimes A_2 u_{N_2})$

$u_{N_2}$ is an eigenvector with eigenvalue 1.
||Ax|| = ||A_1 y|| · ||u_{N_2}|| (operator norm is multiplicative on tensor product)

The matrix shrinks the $L_2$ norm of the vector by its second largest eigenvalue, so we have $\lambda_G ||y|| · ||u_{N_2}|| = \lambda_G ||x||$

Now we consider $||Ax^\perp||$.

Each row of $x^\perp$ is perpendicular to the all-ones vector. If $A_2$ acts on $x^\perp$ it will shrink each row by $\lambda$ (i.e. $||A_2(x^\perp)_1|| \leq \lambda_G ||(x^\perp)_1||$).

$||Ax^\perp|| = (A_1 \otimes A_2)x^\perp = (A_1 \otimes I_{N_2})(I_{N_1} \otimes A_2)x^\perp$ because the matrices are of the right dimension, so we can use the tensor property that we discussed earlier.

How does $(I_{N_1} \otimes A_2)$ act on $x^\perp$? Each row will be $A_2$ times the corresponding row. It shrinks each row of $x^\perp$ by $\lambda$ (i.e. $||A_2(x^\perp)_1|| \leq \lambda_G ||(x^\perp)_1||$).

$||Ax^\perp|| = \lambda_G ||A_1|| \cdot ||x^\perp|| \leq 1 \leq \lambda_G ||x^\perp||$

We have finished both the calculations, so we will finish the proof now. In one term we get $\lambda_G$, and in the other we get $\lambda_G$.

$||Ax||$ is equal to $||A(x^\parallel + x^\perp)||$. By the triangle inequality, $||A(x^\parallel + x^\perp)|| \leq ||Ax^\parallel|| + ||Ax^\perp|| \leq \lambda_G ||x^\parallel|| + \lambda_G ||x^\perp|| \leq (\lambda_G + \lambda_G) ||x||$.

But this is a worse bound than we promised. We promised max, not sum. In order to get a better bound, we observe that, if we can show that $Ax^\parallel$ and $Ax^\perp$ are orthogonal vectors, we can use the Pythagorean Theorem instead of the triangle inequality to get a stronger bound.

Claim 8.1 $Ax^\parallel$ and $Ax^\perp$ are orthogonal vectors.

Proof:

$Ax^\perp$ is perpendicular to $u_{N_2}$ on each cloud because, in the expression $(A_1 \otimes I_{N_2})(I_{N_1} \otimes A_2)x^\perp$, the application of $A_2$ keeps the vector perpendicular, and the application of $A_1$ replaces each cloud with a linear combination of clouds, which also preserves the orthogonality.

$Ax^\parallel$ remains parallel to $u_{N_2}$ on each cloud because $x^\parallel = y \otimes u_{N_2}Ax^\parallel = (A_1 \otimes A_2)(y \otimes u_{N_2}) = (A_1y \otimes u_{N_2})$.

Thus, $Ax^\parallel$ and $Ax^\perp$ are orthogonal vectors.$\blacksquare$

We can now give the desired stronger bound using the orthogonality of the two vectors:

$||Ax^\parallel||^2 = ||Ax^\parallel||^2 + ||Ax^\perp||^2 \leq \lambda_G^2 ||x^\perp||^2 + \lambda_G^2 ||x^\parallel||^2 \leq \max\{\lambda_G, \lambda_G\}^2 (||x^\parallel||^2 + ||x^\perp||^2) = \max\{\lambda_G, \lambda_G\}^2 ||x||^2$

$||Ax|| \leq \max\{\lambda_G, \lambda_G\} ||x||$

8.5 Concluding Remarks

We will cover the zigzag product in the next class.