CS 6815 Pseudorandomness and Combinatorial Constructions **Fall 2019**

Lecture 5: September 12

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5.1 Expander Mixing Lemma

Lemma 5.1 Let $G = (N, \gamma)$ be a spectral expander. $\forall S, T \subseteq [N]$, we have two equivalent equations

$$\left|E(S,T) - \frac{D|S||T|}{N}\right| \leq \lambda\sqrt{|S||T|(1 - \frac{|S|}{N})(1 - \frac{|T|}{N})}$$
$$\left|\frac{E(S,T)}{ND} - \alpha\beta\right| \leq \lambda\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$$

with $\frac{|S|}{N} = \alpha, \frac{|T|}{N} = \beta$. Note that the last two terms under the square root may be omitted for sufficiently small α, β (or equivalently, for sufficiently large $\frac{|S|}{N}, \frac{|T|}{N}$).

Proof: Recall that $E(S,T)$ counts each edge between S and T twice, $\vec{1} = (1,..,1) \in \mathbb{R}^n$, $\vec{\mathbb{1}}_S(i) =$ the indicator vector for $S \in \mathbb{R}^n$, A the normalized adjacency matrix of G.
We see that $\vec{\mathbb{1}}_S^\top A\vec{\mathbb{1}}_T = \frac{E(S,T)}{D} = \sum A_{ij}\vec{\mathbb{1}}_S(i)\vec{\mathbb{1}}_T(j)$ (we can switch i and j since A is symmetric).
Using vector decomposition, we get that $\vec{\mathbb{1}}_S = |S|\vec{1} + \vec{\mathbb{1}}_S^\perp = \sum_{i=1}^n \mu_i\vec{v}_i$, $\vec{\mathbb{1}}_T = |T|\vec{1} + \vec{\mathbb{1}}_T^\perp = \sum_{i=1}^n \rho_i\vec{v}_i$, and ultimately that $\vec{\mathbb{1}}_S A\vec{\mathbb{1}}_T = (|S|\vec{1} + \vec{\mathbb{1}}_S^\perp)^\top A(|T|\vec{1} + \vec{\mathbb{1}}_T^\perp)$.
Note that $\sum \mu_i{}^2 = |S|$, $\sum \rho_i{}^2 = |T|$, $\vec{v}_i = \frac{1}{\sqrt{n}}\vec{1}$, $\mu_1 =< \vec{\mathbb{1}}_S, \vec{v}_1 >= \frac{|S|}{\sqrt{N}}$, $\rho_1 =< \vec{\mathbb{1}}_S, \vec{v}_1 >= \frac{|T|}{\sqrt{N}}$. Putting all this together we see that

$$\begin{aligned}\vec{\mathbb{1}}_S^\top A\vec{\mathbb{1}}_T &= (\sum \mu_i\vec{v}_i)^\top A(\sum \rho_i\vec{v}_i)\\ &= (\sum \mu_i\vec{v}_i)^\top(\sum \rho_i\lambda_i\vec{v}_i)(\vec{v}_i \text{ are eigenvectors})\\ &= \sum_{i=1}^n \mu_i\lambda_i\rho_i(\vec{v}_i \text{ are orthonormal})\end{aligned}$$

This means that $\left|\frac{E(S,T)}{D} - \mu_1 e_1\right| \leq \left|\sum_{i=2}^n \mu_i\lambda_i\rho_i\right| \leq \lambda\left|\sum \mu_i\rho_i\right|$, where $\mu_1 e_1 = \frac{|S||T|}{N}$, $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$.
We invoke the Cauchy-Schwartz inequality to bound the previous by

$$\begin{aligned}\lambda(\sum_{i=2}^n \mu_i^2)^{\frac{1}{2}}(\sum_{i=2}^n \rho_i^2)^{\frac{1}{2}} &\leq \lambda(|S| - \frac{|S|^2}{N})^{\frac{1}{2}}(|T| - \frac{|T|^2}{N})^{\frac{1}{2}}\\ &= \lambda(|S||T|(1-\alpha)(1-\beta))^{\frac{1}{2}}\end{aligned}$$

5.2 Spectral Expansion Implies Vertex Expansion

Proof: Let G be a(N, γ) spectral expander. $\forall \epsilon > 0, G$ is an $(\epsilon N, A)$ vertex expander, where $A = \frac{1}{(1-\epsilon)\lambda^2 + \epsilon}$, $\lambda = 1 - \gamma$. Fix $\epsilon > 0$, $S \subseteq [N]$, $|S| = \epsilon N$, $T = [N] \backslash N(S)$. Suppose $N(S) < A|S| = A\epsilon N$; we observe that $\beta = \frac{|T|}{N} \geq 1 - A\epsilon$. $E(S,T) = 0$, so $\alpha\beta \leq \lambda\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$ by the Expander Mixing Lemma. $\alpha\beta \leq \lambda^2(1-\alpha)(1-\beta)$; $\beta \leq \frac{\lambda^2(1-\alpha)}{\alpha+\lambda^2(1-\alpha)}$. Invoking the earlier bound on beta yields $A\epsilon \geq \frac{\alpha}{\alpha+\lambda^2(1-\alpha)}$, but epsilon is equal to alpha by definition, so $A \geq \frac{1}{\epsilon+\lambda^2(1-\alpha)}$.