17.1 Introduction

Today’s lecture discusses explicit vertex expanders from list-decodable codes and introduces Parvaresh-Vardy codes.

17.2 Vertex Expanders

Recall a $D$-left regular bipartite graph $G$ is a $(K, A)$-bipartite vertex expander with parts $L$ and $R$ ($|L| = N, |R| = M$) if $\forall S \subset L$ with $|S| = K, |\Gamma(S)| \geq AK$.

We know there are probabilistic bounds for these sizes, namely $A = (1 - \epsilon)D, D = O\left(\frac{\log(N/M)}{\epsilon}\right)$, and $M = O\left(\frac{KD}{\epsilon}\right)$. Furthermore, spectral methods do not go beyond $A \sim D/2$ or $M \ll N$ achieving $D \sim \log(N)$.

Note: our discussion will focus on unbalanced expanders (lots of nodes on the left and few on the right).

17.3 Graphs from Codes and List View of Expanders

Given a $(\rho, W)$-list decodable code $C : [N] \rightarrow [M]^D$, we can construct a corresponding bipartite graph in the following way. Let $L = [N]$ and $R = M \times [D]$. Then, for $x \in L$, we add an edge from $x$ to $(i, C(x)_i)$ for $1 \leq i \leq D$. In other words, $\Gamma(x) = \{(i, C(x)_i) : i \in [D]\}$. Notice that this graph is left $D$-regular. See the figure below for an illustration.

We can also consider the list view of expanders. For any $T \subset R$, define $List(T) = \{x \in L : \Gamma(x) \subset T\}$. Similarly, for $\epsilon > 0$, $List(T, \epsilon) = \{x \in L : |\Gamma(x) \cap T| \geq \epsilon D\}$. Note: $List(T) = List(T, 1)$.

Claim 17.1. Let $\lambda = (\lambda_1, \ldots, \lambda_D)$ be a received corrupted word. If $T_\lambda = \{(i, \lambda_i) : i \in [D]\}$, then $|List(T_\lambda, \rho)| \leq W$.

Proof: If $x \in List(T_\lambda, \rho)$, that means $\Delta(C(x), \lambda) \leq (1 - \rho)D$. Since $C$ is $(\rho, W)$-list decodable, there are at most $W$ such $x$’s.

Remark: in general, we can’t say much about $List$.

On the other hand, $G$ is a $(K, A)$ $D$-regular bipartite vertex expander $\iff$ for any $T \subset R, |T| \leq AK - 1, |List(T)| \leq K - 1$. (Observe that the latter is simply the contrapositive of the former.)
*We need to bound List($T, 1$) for all "small" $T$s but LD codes bound "structured" $T$s.
*For expanders, we really care about exact size of List($T, 1$) (even constants matter!!). But for list decodable codes, exact size of List does not matter.

### 17.4 Parvaresh-Vardy Codes

Fix a finite field $\mathbb{F}_q$ and message space $f \in Poly_{\leq n-1}$ [univariate polynomials over $\mathbb{F}_q$ with degree $\leq n - 1$]. The encoding will map messages from $\mathbb{F}_q^n$ to $\mathbb{F}_q^m$ with $C \subset \Sigma^n, |\Sigma| = q^m = |\mathbb{F}_q^m|$. Intuitively, we’re taking a polynomial $f$ and sending it to a set of polynomials $f_1, \ldots, f_m$ and evaluating each $f_i$ at all points in $\mathbb{F}_q$.

Let $E(x)$ be an irreducible polynomial of degree $n$ over $\mathbb{F}_q$. Consider the extension field $F = \mathbb{F}_q[x]/E(x)$. Think of $f \in F$ and compute $f_i = (f)^{h^i}$, $i = 0, 1, \ldots, m - 1$ (note $h$ is not yet set). Also, note $f_0 = f$, and we can think of each $f_i \in Poly_{\leq n-1}$.

Then, the list-decoding radius of PV code (with appropriate choice of parameters) is $1 - r^{2/3}$ (where $r$ is the relative rate). Recall for RS it’s $1 - r^{1/2}$.

We can now consider the graph $G$ from the PV code. We have $L = \mathbb{F}_q^n = \mathbb{F}_q^{\leq n-1}$ and $\Gamma(f, y) = [y, f_0(y), \ldots, f_{m-1}(y)]$ where $f \in Poly_{\leq n-1}$ and $y \in \mathbb{F}_q$. Note that $G$ is a $q$-left regular graph.

**Theorem 17.2** $G$ is a $(K = h^m, A = q - (n - 1)(h-1)m)$ vertex expander.
We will see the proof next class.

The takeaway is that we can construct a highly-unbalanced graph with near-optimal expansion.

If we’re given $N, K, \epsilon, \alpha > 1$, then we define $n = \log_2(N), k = \log_2(K), h = (nk/\epsilon)^{1/\alpha}, q \in (h^{1+\alpha}, 2h^{1+\alpha})$ a power of 2, and $m = \log_h(K)$. Then, $|L| = q^n \geq N, |R| = q^{m+1} \leq q^2K^{1+\alpha}, D = q \leq O\left(\frac{\log(N)\log(K)}{\epsilon}\right)^{1+1/\alpha}$ and $A \geq (1-\epsilon)q$. 