In this lecture, we will prove that a list-decodable code is also a strong seeded extractor:

**Theorem 1.** Let $C : [N] \to [M]^D$ be a $(1 - \frac{1}{M} - \epsilon, L)$ list decodable code. Then $\text{Ext} : [N] \times [D] \to [M]$ defined by

$$\text{Ext}(x, y) = C(x)_y$$

is a strong seeded extractor for min-entropy $k = \log L + \log(1/\epsilon)$ with error $2M\epsilon$. 

We have seen that a $(k, \epsilon)$ strong seeded extractor $\text{Ext} : [N] \times [D] \to [M]$ can be represented by a left-$D$-regular bipartite graph $\text{Ext} = ([N], [M] \times [D])$ such that $x \in [N]$ in the left vertex set is connected to $(y, \text{Ext}(x, y))$ in the right vertex set for every edge label $y \in [D]$. For a list decodable code $C : [N] \to [M]^D$, we can similarly define a left-$D$-regular bipartite graph $C = ([N], [M] \times [D])$ such that a vertex $x \in [N]$ is connected to $(y, C(x)_y)$ for every $y \in [D]$. We define the following notation for left-regular bipartite graph:

**Definition 1.** For left-$D$-regular bipartite graph $G = (L, R)$, $T \subseteq R$ and parameter $\delta \in [0, 1]$, define

$$\text{LIST}_G(T, \delta) = \{ x \in L \mid |\Gamma(x) \cap T| \geq \delta |T| \}.$$

By definition of list-decodable code, we have the following lemma:

**Lemma 1.** Let $C : [N] \to [M]^D$ be a $(1 - \delta, L)$ list-decodable code, and $T = \{ (y, z) \mid y \in [D] \}$ for any $(z_1, z_2, \ldots, z_D) \in [M]^D$. Then $|\text{LIST}_C(T, \delta) \leq L|$. 

Now we are ready to prove the theorem.

**Proof of Theorem 1.** Consider the bipartite graph $C = ([N], [M] \times [D])$. Let $X$ be a subset of $[N]$ of size $K$. ($K$ will be specified later.) Observe that for the $k$-source $U_X$ uniformly distributed over $X$, uniformly random $Y \in [D]$ and every $y \in [D]$, $z \in [M]$,

$$\Pr[(Y, C(U_X)|_Y) = (y, z)] = \frac{|\Gamma((y, z)) \cap X|}{KD}.$$

Then the statistical distance between $(Y, C(U_X)|_Y)$ and uniform distribution is

$$\sum_{y \in [D], z \in [M]} \max \left( \frac{|\Gamma((y, z)) \cap X|}{KD} - \frac{1}{MD} \cdot 0 \right).$$

For every $y \in [D]$, define $z_y = \arg \max_{z \in [M]} |\Gamma((y, z)) \cap X|$. Note that $|\Gamma((y, z_y)) \cap X| \geq K/M$ by averaging.

Let $T = \{(y, z_y) \mid y \in [D]\}$ and $\delta = 1/M + \epsilon$. Then

$$\sum_{y \in [D], z \in [M]} \max \left( \frac{|\Gamma((y, z)) \cap X|}{KD} - \frac{1}{MD} \cdot 0 \right) \leq \sum_{y \in [D]} \left( \frac{M \cdot |\Gamma((y, z_y)) \cap X|}{KD} - \frac{1}{D} \right) \leq \frac{M \cdot |\text{LIST}_C(T, \delta)| \cdot D + K \cdot \delta D}{KD} - 1 = M\epsilon + \frac{ML}{K}.$$

Choose $K = L/\epsilon$ we can conclude that $C$ is a $(\log L + \log(1/\epsilon), 2M\epsilon)$ extractor. \(\square\)