COM S 6810 Theory of Computing January 20, 2009

Handout 1: Notation and Probability

Instructor: Rafael Pass *Teaching Assistant: Muthu Venkitasubramaniam*

Notation

Algorithm

Let $\mathcal{A}$ denote an algorithm. We write $\mathcal{A}(.)$ to denote an algorithm with one input and $\mathcal{A}(.,.)$ for two inputs. In general, the output of an algorithm can be considered as a probability distribution. So $\mathcal{A}(x)$ denotes a probability distribution. The algorithm is deterministic if the probability is concentrated on a single element.

Experiment

To sample an element x from a distribution S we denote the experiment by $x \leftarrow S$. If F is a finite set, then $x \leftarrow \mathcal{F}$ is the experiment of sampling uniformly from the set F. To denote the ordered sequence in which the experiments happen we use semicolon.

$$(x \leftarrow S; (y, z) \leftarrow A(x))$$

Using this notation we can describe probability of events. If $p(.,.)$ denotes a predicate, then

$$Pr[x \leftarrow S; (y, z) \leftarrow A(x) : p(y, z)]$$

is the probability that the predicate p(y,z) is true after the ordered sequence of events $(x \leftarrow S;\ (y, z) \leftarrow A(x))$. The notation $\{x \leftarrow S; (y, z) \leftarrow A(x) : (y, z)\}$ denotes the probability distribution $\{y, z\}$ generated by the ordered sequence of experiments $(x \leftarrow S; (y, z) \leftarrow A(x))$.

Probability

Basic Facts

- Events A and B are said to be *independent* if

$$Pr[A \cap B] = Pr[A] \cdot Pr[B]$$

- Events $A_1, A_2, \ldots, A_n$ are said to be *pairwise independent* if for every i and every $j \neq i$, A_i and A_j are independent.
- *Union Bound:* Let $A_1, A_2, \ldots, A_n$ be events. Then,
$$Pr[A_1 \cup A_2 \cup \ldots \cup A_n] \leq Pr[A_1] + Pr[A_2] + \ldots Pr[A_n]$$
- Let X be a random variable with range Ω. The *expectation* of X is a number defined as follows.
$$E[X] = \sum_{x \in \Omega} x Pr[X = x]$$
The *variance* is given by,
$$Var[X] = E[X^2] - (E[X])^2$$
- Let $X_1, X_2, \ldots, X_n$ be random variables. Then,
$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$
- If X and Y are *independent random variables*, then
$$\begin{aligned} E[XY] &= E[X] \cdot E[Y] \\ Var[X + Y] &= Var[X] + Var[Y] \end{aligned}$$

Markov's Inequality

If X is a positive random variable with expectation $E(X)$ and $a > 0$, then
$$Pr[X \geq a] \leq \frac{E(X)}{a}$$

Chebyshev's Inequality

Let X be a random variable with expectation $E(X)$ and variance σ^2, then for any $k > 0$,
$$Pr[|X - E(X)| \geq k] \leq \frac{\sigma^2}{k^2}$$

Chernoff's inequality

Let $X_1, X_2, \ldots, X_n$ denote independent random variables, such that for all i, $E(X_i) = \mu$ and $|X_i| \leq 1$.
$$Pr\left[\left|\frac{\sum X_i}{n} - \mu\right| \geq \epsilon\right] \leq 2^{-\epsilon^2 n}$$