Machine Learning Theory (CS 6783)

1 Two Equivalent Definitions of Convexity

For this section, say $\mathcal{B}$ is some vector space equipped with norm $\|\cdot\|$ and $\mathcal{B}^*$ be the dual space equipped with dual norm $\|\cdot\|_*$. For simplicity think of $\mathcal{B}$ and $\mathcal{B}^*$ to simply be $\mathbb{R}^d$. The following are two equivalent definitions of convex functions.

Definition 1. *A function $f : \mathcal{B} \mapsto \mathbb{R}$ is said to be convex if for all $x, y \in \mathcal{B}$ and any $\alpha \in [0, 1]$,*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

The following is an equivalent definition in terms of gradients.

Definition 2. *A function $f : \mathcal{B} \mapsto \mathbb{R}$ is said to be convex if for all $x, y \in \mathcal{B}$*

$$f(x) \leq f(y) + \langle \nabla f(x), x - y \rangle$$

Why are the two definitions equivalent?
(**1** $\Rightarrow$ **2**) First lets show that the first definition implies the second. Note that by definition of directional derivative:

$$\begin{aligned}
\langle \nabla f(x), y - x \rangle &= \lim_{\alpha \to 0} \frac{f(x + \alpha(y - x)) - f(x)}{\alpha} \\
&= \lim_{\alpha \to 0} \frac{f((1-\alpha)x + \alpha y)) - f(x)}{\alpha} \\
&\leq \lim_{\alpha \to 0} \frac{(1-\alpha) f(x) + \alpha f(y) - f(x)}{\alpha} \\
&= \lim_{\alpha \to 0} \frac{\alpha(f(y) - f(x))}{\alpha} \\
&= f(y) - f(x)
\end{aligned}$$

Rearranging we see that definition 1 implies definition 2.

(**2** $\Rightarrow$ **1**) Now to prove the other direction, starting with definition 2, we have the following two inequalities:

$$\begin{aligned}
f(\alpha x + (1-\alpha)y) &\leq f(x) + \langle \nabla f(\alpha x + (1-\alpha)y), \alpha x + (1-\alpha)y - x \rangle \\
&= f(x) + (1-\alpha) \langle \nabla f(\alpha x + (1-\alpha)y), y - x \rangle
\end{aligned}$$

Similarly,

$$
\begin{aligned}
f(\alpha x + (1-\alpha) y) &\leq f(y) + \langle \nabla f(\alpha x + (1-\alpha) y), \alpha x + (1-\alpha) y - y \rangle \\
&= f(y) + \alpha \langle \nabla f(\alpha x + (1-\alpha) y), y - x \rangle
\end{aligned}
$$

Hence summing up α times the first inequality and $1-\alpha$ times the second we end up with

$$
\begin{aligned}
f(\alpha x + (1-\alpha) y) &\leq \alpha f(x) + (1-\alpha) f(y) + (1-\alpha)\alpha \langle \nabla f(\alpha x + (1-\alpha) y), y - x \rangle + \\
&\quad alpha(1-\alpha) \langle \nabla f(\alpha x + (1-\alpha) y), x - y \rangle \\
&= \alpha f(x) + (1-\alpha) f(y)
\end{aligned}
$$

This completes the proof.