1 Stability of $\phi$ of Specific Form

Let $F \subseteq \{\pm 1\}^n$ be a set of $n$ dimensional $\pm 1$ vectors. One form of $\phi$ we might be interested in for Cover’s result is

$$\phi(y_1, \ldots, y_n) = \frac{1}{n} \min_{f \in F} \sum_{t=1}^{n} 1\{y_t \neq f_t\} + C_n(F)$$

That is, we want our average error to be bounded by average classification error of the best $f \in F$ (chosen in hindsight of course) + an extra slack of $C_n(F)$. To apply Cover’s result from class, we need two things. First, that $\phi$ is stable and next, to be able to provide guarantee against any arbitrary adversary, we also need that $E[\phi(e_1, \ldots, e_n)] \geq 1/2$.

Before we focus on these two properties, note that when $F = \{(+1, \ldots, +1), (-1, \ldots, -1)\}$, then $\frac{1}{n} \min_{f \in F} \sum_{t=1}^{n} 1\{y_t \neq f_t\}$ is exactly the average error of going with majority in hindsight and we already argued that for this case $C_n(F) = \Theta(1/\sqrt{n})$ to ensure that $E[\phi(e_1, \ldots, e_n)] \geq 1/2$.

Moving to the general case, first towards the question of stability we have the following claim.

Claim 1. For any $F \subseteq \{\pm 1\}^n$ if we consider any $\phi$ defined by

$$\phi(y_1, \ldots, y_n) = \frac{1}{n} \min_{f \in F} \sum_{t=1}^{n} 1\{y_t \neq f_t\} + C_n(F),$$

such a $\phi$ is stable.

Proof. Consider any $i \in [n]$ and any $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$. Without loss of generality, consider the case when $\phi(y_1, \ldots, y_{i-1}, +1, y_{i+1}, \ldots, y_n) \geq \phi(y_1, \ldots, y_{i-1}, -1, y_{i+1}, \ldots, y_n)$ because when it is not the case we simply consider the same argument below for the terms in the difference swapped.

$$\phi(y_1, \ldots, y_{i-1}, +1, y_{i+1}, \ldots, y_n) - \phi(y_1, \ldots, y_{i-1}, -1, y_{i+1}, \ldots, y_n)$$

$$= \frac{1}{n} \left( \min_{f \in F} \left( \sum_{t=1}^{i-1} 1\{y_t \neq f_t\} + 1\{+1 \neq f_i\} + \sum_{t=i+1}^{n} 1\{y_{t} \neq f_t\} \right) - \min_{f \in F} \left( \sum_{t=1}^{i-1} 1\{y_{t} \neq f_t\} + 1\{-1 \neq f_i\} + \sum_{t=i+1}^{n} 1\{y_t \neq f_t\} \right) \right)$$

Noting that $-\min(\ldots)$ is same as $\max(-\ldots)$, and rearranging we get,

$$= \frac{1}{n} \max_{f \in F} \left( \min_{f \in F} \left( \sum_{t=1}^{i-1} 1\{y_{t} \neq f_t\} + 1\{+1 \neq f_i\} + \sum_{t=i+1}^{n} 1\{y_{t} \neq f_t\} \right) - \sum_{t=1}^{i-1} 1\{y_t \neq f'_t\} - 1\{-1 \neq f'_i\} - \sum_{t=i+1}^{n} 1\{y_t \neq f'_t\} \right)$$
replacing the minimum with \( f' \) as the choice to use we get,

\[
\leq \frac{1}{n} \max_{f' \in \mathcal{F}} \left( \sum_{i=1}^{n-1} 1\{y_i \neq f'_i\} + 1\{+1 \neq f'_i\} + \sum_{i+1}^{n} 1\{y_i \neq f'_i\} - \sum_{i=1}^{n-1} 1\{y_i \neq f'_i\} - 1\{-1 \neq f'_i\} - \sum_{i+1}^{n} 1\{y_i \neq f'_i\} \right)
\]

\[
= \frac{1}{n} \max_{f' \in \mathcal{F}} \{1\{+1 \neq f'_i\} - 1\{-1 \neq f'_i\}\}
\]

\[
\leq \frac{1}{n}
\]

Hence, \( \phi \) is stable for any \( \mathcal{F} \).

Now that we have established that \( \phi \) is stable, lets move to what the minimum value of \( C_n(\mathcal{F}) \) should be to ensure that \( \mathbb{E}_\epsilon [\phi(\epsilon)] \geq 1/2 \). To this end, clearly the smallest \( C_n(\mathcal{F}) \) is the one where we have the equality

\[
\frac{1}{2} = \mathbb{E}_\epsilon [\phi(\epsilon)] = \frac{1}{n} \mathbb{E}_\epsilon \left[ \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{t=1}^{n} 1\{\epsilon_t \neq f_t\} \right] + C_n(\mathcal{F})
\]

and so we have that

\[
C_n(\mathcal{F}) = \frac{1}{2} - \mathbb{E}_\epsilon \left[ \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{t=1}^{n} 1\{\epsilon_t \neq f_t\} \right]
\]

using that fact that for any \( y, y' \in \{\pm 1\} \), \( 1\{y \neq y'\} = (1 - y \cdot y')/2 \) we get,

\[
= \frac{1}{2} - \mathbb{E}_\epsilon \left[ \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{t=1}^{n} \frac{1}{2} (1 - \epsilon_t \cdot f_t) \right]
\]

\[
= \frac{1}{2} - \mathbb{E}_\epsilon \left[ \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{t=1}^{n} \left( \frac{1}{2} - \frac{1}{2} \epsilon_t \cdot f_t \right) \right]
\]

\[
= \frac{1}{2} - \mathbb{E}_\epsilon \left[ \min_{f \in \mathcal{F}} \left( \frac{1}{2} - \frac{1}{2n} \sum_{t=1}^{n} \epsilon_t \cdot f_t \right) \right]
\]

\[
= -\mathbb{E}_\epsilon \left[ \min_{f \in \mathcal{F}} \left( -\frac{1}{2n} \sum_{t=1}^{n} \epsilon_t \cdot f_t \right) \right]
\]

\[
= \frac{1}{2n} \mathbb{E}_\epsilon \left[ \max_{f \in \mathcal{F}} \sum_{t=1}^{n} \epsilon_t \cdot f_t \right]
\]

\[
= \frac{1}{2n} \mathbb{E}_\epsilon \left[ \max_{f \in \mathcal{F}} f^\top \epsilon \right]
\]

where in the last line we can thing of \( f \) and \( \epsilon \) as \( n \) dimensional vectors.