**ATLAS BACKFLIP**

**State:** Joint Angles / Angular Velocity
Base Pose

**Action:** Joint Torques

**Transition:** Physics

**Cost:** Renormalization of Errors

- $J^2(x, u)$
- "Trajectory + Parabola"

Rotate
**Goal:** Build a controller for an inverted pendulum (MDP route)

**Define MDP for a pendulum**

\[
\text{Torque} = mg l \sin \theta + \tau = I \ddot{\theta} = ml^2 \ddot{\theta}
\]

\[
\dot{\theta} = \frac{g}{l} \sin \theta + \frac{\tau}{ml^2}
\]

**Euler-Lagrange**

\[
L = T - V
\]

\[
= \frac{1}{2} ml^2 \dot{\theta}^2 - mgl \cos \theta
\]

\[
\frac{d}{dt} \left( \frac{2L}{\dot{\theta}} \right) - \frac{2L}{\dot{\theta}} = \tau
\]

\[
\frac{d}{dt} \left( ml^2 \dot{\theta} \right) + mgl \sin \theta = \tau
\]

\[
ml^2 \dddot{\theta} + mgl \sin \theta = \tau
\]

**Acrobat**
Let's linearize about \((\theta = 0, \dot{\theta} = 0)\)

\[
\dot{\theta} = \frac{g}{l} \sin \theta + \frac{\gamma}{m l^2}
\]

\[
\dot{\theta} \approx \frac{g}{l} \theta + \frac{\gamma}{m l^2}
\]

(Dynamics)

State:

\[
X_t = \begin{bmatrix} \theta_t \\ \dot{\theta}_t \end{bmatrix}
\]

Action:

\[
u_t = \gamma
\]

Dynamics:

\[
X_{thi} = f(X_t, u_t)
\]

\[
\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}_{thi} = \begin{bmatrix}
1 + \frac{g}{l} (\frac{1}{2} \dot{\theta}^2) \\
\frac{\gamma}{m l^2}
\end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}_t + \frac{1}{m l^2} \left[ \begin{bmatrix} \Delta t^2 \\ \Delta t \end{bmatrix} \right]_{\Delta t = \text{Time step}}
\]

\[
X_{thi} = AX_t + Bu_t
\]
\( C(x, u) = x^T Q x + u^T R u \)

\[
\begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
- \begin{bmatrix}
\theta x_f \\
\dot{\theta} x_{x_f}
\end{bmatrix}
= \begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix} + \omega^2 \theta^2
\]

\[
\mathbf{e} = \begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
= \theta^2 + \dot{\theta}^2 + \omega^2
\]

\[
Q_1 = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
Q_2 = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
C(x, u) = -\theta^2 + \dot{\theta}^2
\]

**Q** must be symmetric.

**R** must be symmetric.

**Positive Semidefinite** \( Q > 0 \)

\( x^T Q x > 0 \)
Value Iteration via Dynamic Programming

\[ V_t(x_t) = \min_{u_t} \left[ c(x_t, u_t) + V_{t+1}(x_{t+1}) \right] \]

\[ V_{t-1}(x_{t-1}) = \min_{u_{t-1}} \left[ c(x_{t-1}, u_{t-1}) + 0 \right] \]

\[ = \min_{u_{t-1}} \left[ x_{t-1}^T Q x_{t-1} + u_{t-1}^T R u_{t-1} \right] \]

\[ = \frac{\partial}{\partial u_{t-1}} \left( x_{t-1}^T Q x_{t-1} + u_{t-1}^T R u_{t-1} \right) = 0 \]

\[ 2 u_{t-1}^T R = 0 \]

\[ u_{t-1} = 0 \]

\[ V_{t-1}(x_{t-1}) = x_{t-1}^T Q x_{t-1} \quad \text{"QUADRATIC"} \]

\[ \Delta = x_{t-1}^T V_{t-1} x_{t-1} \quad \text{"THE TRICK"} \]
Arbitrary timestep $t$

$$V_t(x_t) = \min_{u_t} \left[ c(x_t, u_t) + V_{t+1}(x_{t+1}) \right]$$

$$= \min_{u_t} \left[ x_t^T Q x_t + u_t^T R u_t + x_{t+1}^T V_{t+1} x_{t+1} \right]$$

Substitute $x_{t+1} = A x_t + B u_t$

$$\min_{u_t} \left[ x_t^T Q x_t + u_t^T R u_t + (A x_t + B u_t)^T V_{t+1} (A x_t + B u_t) \right]$$

$$\frac{\partial (\cdot)}{\partial u_t} = 0$$

$$\frac{\partial}{\partial x_t} (A x) = A$$

$$\frac{\partial}{\partial x_t} (x^T A) = A^T$$

$$\frac{\partial}{\partial u_t} \left( (A x_t + B u_t)^T V_{t+1} B \right) = 0$$

$$R u_t + B^T V_{t+1} (A x_t + B u_t) = 0$$

$$R u_t + B^T V_{t+1} A x_t + B^T V_{t+1} B u_t = 0$$

$$(C R + B^T V_{t+1} B) u_t = -B^T V_{t+1} A x_t$$

$$u_t = - \left( C R + B^T V_{t+1} B \right)^{-1} B^T V_{t+1} A x_t$$

$$u_t = - K_t$$
\[ V_t(x_t) = x_t^T \left( Q + \kappa_t^T R \kappa_t + (A+B \kappa_t)^T V_{t+1} (A+B \kappa_t) \right) x_t \]

- \( V_b \)