Lecture 6: Monte Carlo Integration

Chapter 3 in Advanced GI

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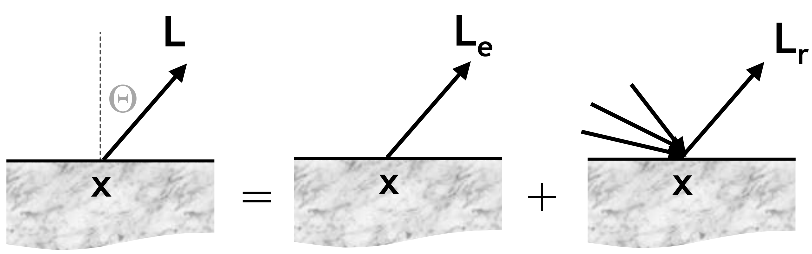
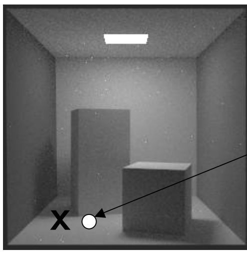
Rendering Equation



$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{hemisphere} L(x \leftarrow \Psi) \, f_r(x, \Psi \leftrightarrow \Theta) \cos(N_x, \Psi) \, d\omega_\Psi$$

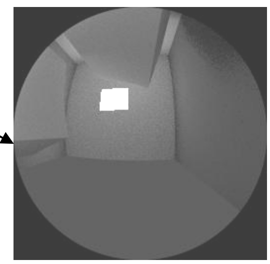
- **Applicable for each wavelength**

Rendering Equation



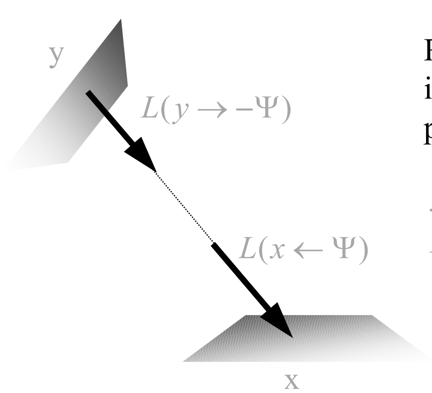
$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{hemisphere} L(x \leftarrow \Psi) f_r(x, \Psi \leftrightarrow \Theta) \cos(\mathbf{N_x}, \Psi) d\omega_\Psi$$

incoming radiance



RE: Area Formulation

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos\theta_x \cdot d\omega_\Psi$$



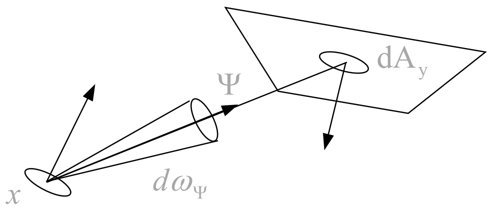
Ray-casting function: what is the nearest visible surface point seen from x in direction Ψ?

$$y = vp(x, \Psi)$$

$$L(x \leftarrow \Psi) = L(vp(x, \Psi) \rightarrow -\Psi)$$

Rendering Equation

$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos\theta_x \cdot d\omega_\Psi$$



$$y = vp(x, \Psi)$$

$$L(x \leftarrow \Psi) = L(vp(x, \Psi) \to -\Psi)$$

$$d\omega_\Psi = \frac{dA_y \cos\theta_y}{r_{xy}^2}$$

Rendering Equation: visible surfaces

$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos\theta_x \cdot d\omega_\Psi$$

Coordinate transform ↓

$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{\substack{y \text{ on} \\ \text{all surfaces}}} f_r(\Psi \leftrightarrow \Theta) \cdot L(y \to -\Psi)\ \cos\theta_x \cdot \frac{\cos\theta_y}{r_{xy}^2} \cdot dA_y$$

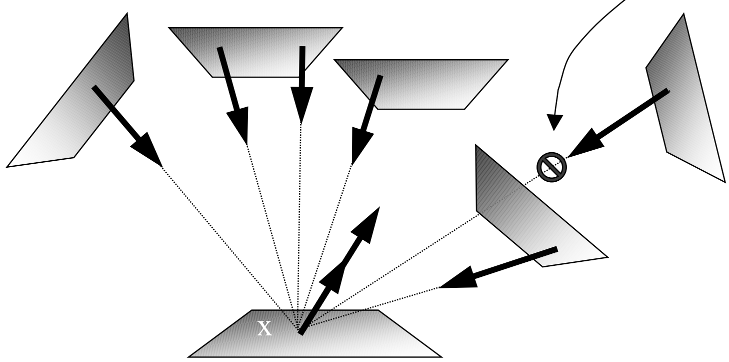
$$y = vp(x, \Psi)$$

Integration domain = visible surface points y

- Integration domain extended to ALL surface points by including visibility function

Rendering Equation: all surfaces

$$L(x \to \Theta) = L_e(\ldots) + \int_A f_r(\ldots) \cdot L(y \to -\Psi) \cdot \frac{\cos\theta_x \cdot \cos\theta_y}{r_{xy}^2} \cdot V(x,y) \, dA_y$$



Two forms of the RE

- Hemisphere integration

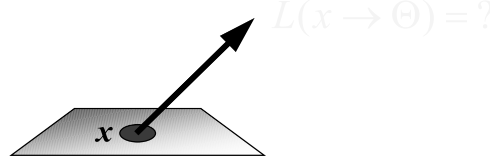
$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos\theta_x \cdot d\omega_\Psi$$

- Area integration (over polygons from set A)

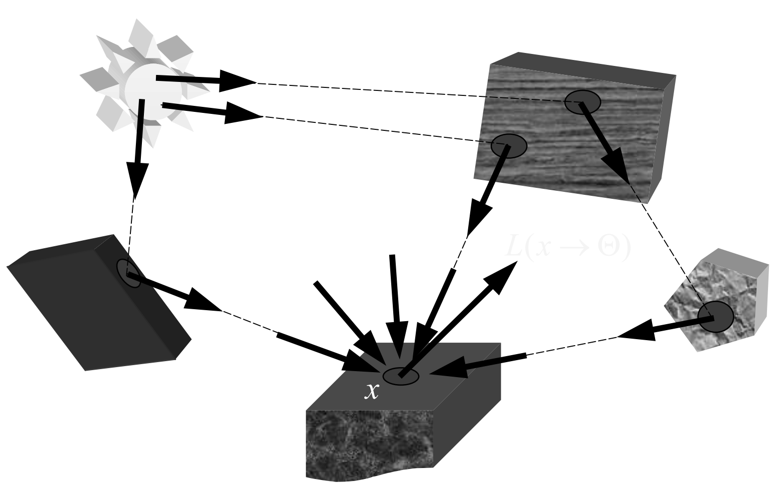
$$L(x \to \Theta) = L_e(x \to \Theta) + \int_A f_r(\Psi \leftrightarrow \Theta) \cdot L(y \to -\Psi) \cdot \frac{\cos\theta_x \cdot \cos\theta_y}{r_{xy}^2} \cdot V(x,y) \cdot dA_y$$

Radiance evaluation

- Fundamental problem in GI algorithms
  - Evaluate radiance at a given surface point in a given direction
  - Invariance defines radiance everywhere else

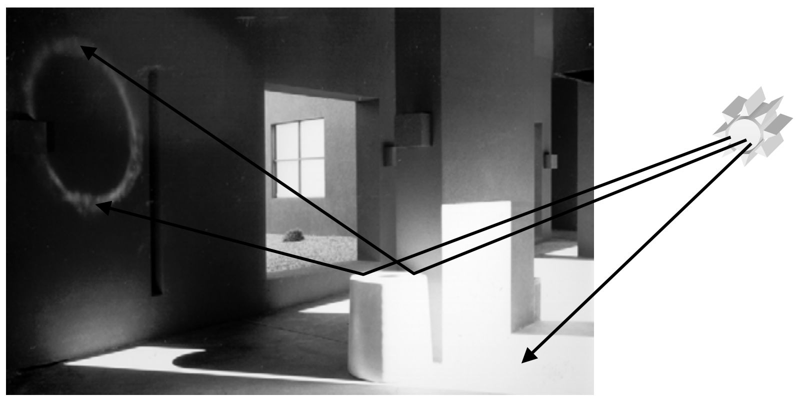


Radiance evaluation



… find paths between sources and surfaces to be shaded

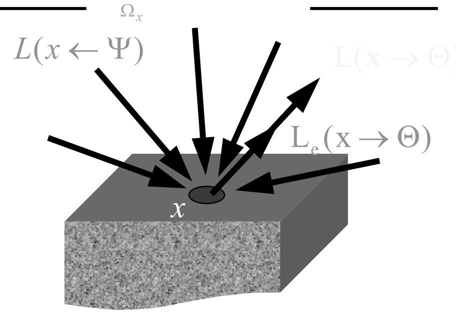
Hard to find paths



Why Monte Carlo?

- Radiance is hard to evaluate

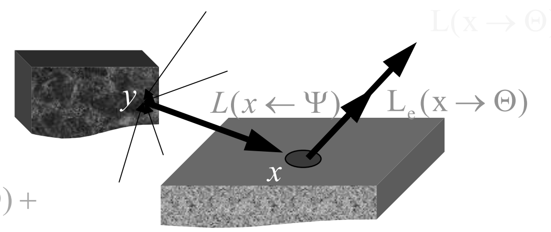
$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



- Sample many paths: integrate over all incoming directions

Why Monte Carlo?

$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



$$L(x \to \Theta) = L_e(x \to \Theta) +$$

$$\int_{\Omega_x} L_e(y \to -\Psi) f_r(\Psi \leftrightarrow \Theta) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi +$$

$$\int_{\Omega_x} \int_{\Omega_y} f_r(\Psi' \to -\Psi) \cos(\Psi', n_y') L(y \leftarrow \Psi') d\omega_{\psi'} f_r(\Psi \leftrightarrow \Theta) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

Why Monte Carlo?

$$L(x \to \Theta) = L_e(x \to \Theta) +$$

$$\int_{\Omega_x} L_e(y \to -\Psi) f_r(\Psi \leftrightarrow \Theta) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi +$$

....

- Analytical integration is difficult
- Therefore, need numerical techniques

Monte Carlo Integration

- Numerical tool to evaluate integrals
- Use sampling
- Stochastic errors
- Unbiased
  – on average, we get the right answer!

Probability

- Random variable x
- Possible outcomes: $x_1, x_2, x_3, \ldots, x_n$
  – each with probability: $p_1, p_2, p_3, \ldots, p_n$
- E.g. 'average die': 2,3,3,4,4,5
  – outcomes: $x_1 = 2, x_2 = 3, x_3 = 4, x_3 = 5$
  – probabilities:

$$p_1 = 1/6, p_2 = 1/3, p_3 = 1/3, p_3 = 1/6$$

Expected value

- Expected value = average value

$$E[x] = \sum_{i=1}^{n} x_i p_i$$

- E.g. die:

$$E[x] = 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 5 \cdot \frac{1}{6} = 3.5$$

Variance

- Expected 'distance' to expected value

$$\sigma^2[x] = E[(x - E[x])^2]$$

- E.g. die:

$$\sigma^2[x] = (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{3} + (4-3.5)^2 \cdot \frac{1}{3} + (5-3.5)^2 \cdot \frac{1}{6}$$
$$= 0.916$$

- Property: $\sigma^2[x] = E[x^2] - E[x]^2$

Continuous random variable

- Random variable $x \in [a,b]$
- Probability density function (pdf) $p(x)$
- Probability that variable has value x: $p(x)dx$

$$\int_a^b p(x)dx = 1$$

- Cumulative distribution function (CDF)
  - CDF is non-decreasing, positive

$$\Pr(x \le y) = CDF(y) = \int_{-\infty}^{y} p(x)dx$$

Continuous random variable

- Expected value: $E[x] = \int_a^b xp(x)dx$

$$E[g(x)] = \int_a^b g(x)p(x)dx$$

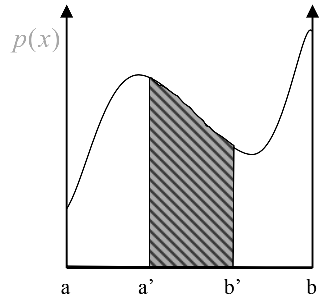
- Variance:

$$\sigma^2[x] = \int_a^b (x - E[x])^2 p(x)dx$$

$$\sigma^2[g(x)] = \int_a^b (g(x) - E[g(x)])^2 p(x)dx$$

- Deviation: $\sigma[x], \sigma[g(x)]$

Continuous random variable



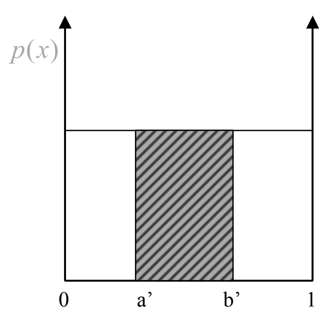
$$\int_a^b p(x)dx = 1$$

$$\Pr(x \le y) = CDF(y) = \int_{-\infty}^{y} p(x)dx$$

Probability that x belongs to [a',b'] $= \Pr(x \le b') - \Pr(x \le a')$

$$= \int_{-\infty}^{b'} p(x)dx - \int_{-\infty}^{a'} p(x)dx = \int_{a'}^{b'} p(x)dx$$

Uniform distribution



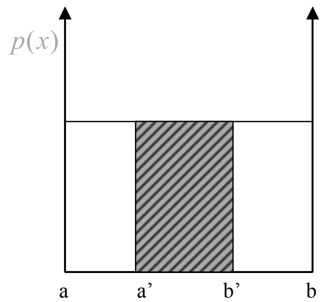
$$\int_a^b p(x)dx = 1$$

$$p(x) = \frac{1}{1-0} = 1$$

$$\Pr(x \in [a',b']) = \int_{a'}^{b'} 1dx = (b'-a')$$

$$\Pr(x \le y) = CDF(y) = \int_{-\infty}^{y} p(x)dx = y$$

Uniform distribution



$$\int_a^b p(x)dx = 1$$

$$p(x) = \frac{1}{b-a}$$

Probability that x belongs to [a',b'] $= \int_{a'}^{b'} \frac{1}{(b-a)} dx = \frac{(b'-a')}{(b-a)}$

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x)dx = \frac{(y-a)}{(b-a)}$$

More than one sample

- Consider the weighted sum of N samples
- Expected value $E[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)] = E[x]$
- Variance $\sigma^2[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)] = \frac{1}{N}\sigma^2[x]$
- Deviation $\sigma[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)] = \frac{1}{\sqrt{N}}\sigma[x]$

More than one sample

- Consider the weighted sum of N samples

$$g(x) = \frac{1}{N}(f(x_1) + f(x_2) + f(x_3) + \ldots f(x_N))$$

- Expected value

$$E[g(x)] = E[\frac{1}{N}\sum_i^N f(x_i)] = E[f(x)]$$

- Variance

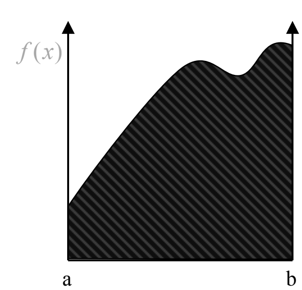
$$\sigma^2[g(x)] = \sigma^2[\frac{1}{N}\sum_i^N f(x_i)] = \frac{1}{N}\sigma^2[f(x)]$$

- Deviation $\sigma[g(x)] = \frac{1}{\sqrt{N}}\sigma[f(x)]$

Numerical Integration

- A one-dimensional integral:

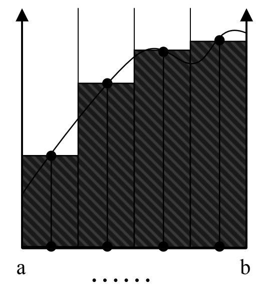
$$I = \int_a^b f(x)dx$$



Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)dx$$
$$\approx \sum_{i=1}^{N} w_i f(x_i)$$

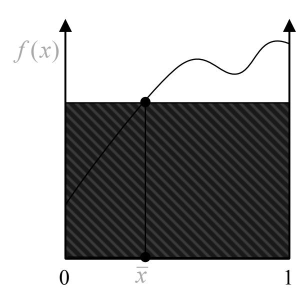


Monte Carlo Integration

Primary estimator:

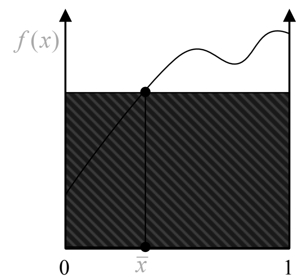
$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$



Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

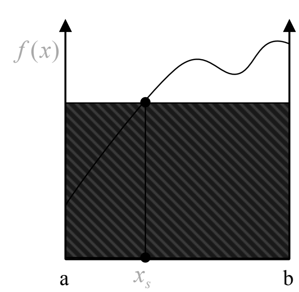
$$I_{prim} = f(\bar{x})$$

$$E(I_{prim}) = \int_0^1 f(x)p(x)dx = \int_0^1 f(x)1dx = I$$

Unbiased estimator!

Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(x_s)(b-a)$$

$$E(I_{prim}) = \int_a^b f(x)(b-a)p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

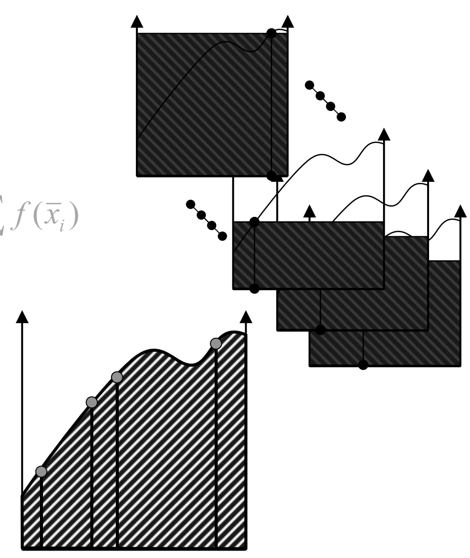
$$\sigma^2_{prim} = \int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x) dx$$

More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{\sec} = \frac{1}{N} \sum_{i=1}^{N} f(\overline{x}_i)$



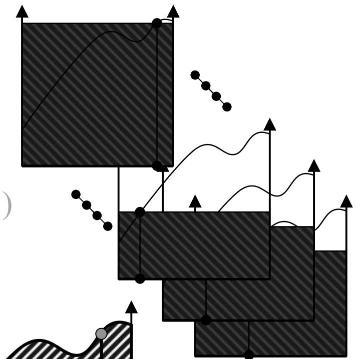
Variance $\sigma^2_{\sec} = \sigma^2_{prim} / N$

More samples

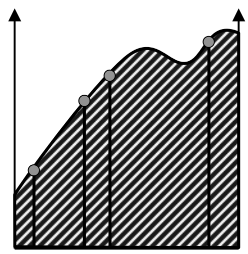
Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{\sec} = \frac{1}{N}\sum_{i=1}^{N} f(x^s{}_i)(b-a)$



Variance $\sigma^2_{\sec} = \sigma^2_{prim} / N$



Monte Carlo Integration

- Expected value of estimator

$$E[\langle I \rangle] = E[\frac{1}{N}\sum_i^N \frac{f(x_i)}{p(x_i)}] = \frac{1}{N}\int (\sum_i^N \frac{f(x_i)}{p(x_i)}) p(x)dx$$

$$= \frac{1}{N}\sum_i^N \int (\frac{f(x)}{p(x)}) p(x)dx$$

$$= \frac{N}{N}\int f(x)dx = I$$

  - on ‘average’ get right result: **unbiased**

- Standard deviation σ is a measure of the stochastic error

$$\sigma^2 = \frac{1}{N}\int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

Convergence Rates

- RMS error converges at a rate of O($\frac{1}{\sqrt{N}}$)
- Unbiased
- Chebychev's inequality

$$\Pr\left[\left|F - E(F)\right| \geq \sqrt{\frac{1}{\delta}}\sigma\right] \leq \delta$$

$$\Pr\left[\left|I_{estimator} - I\right| \geq \frac{1}{\sqrt{N}}\sqrt{\frac{1}{\delta}}\sigma\right] \leq \delta$$

- Strong law of large numbers

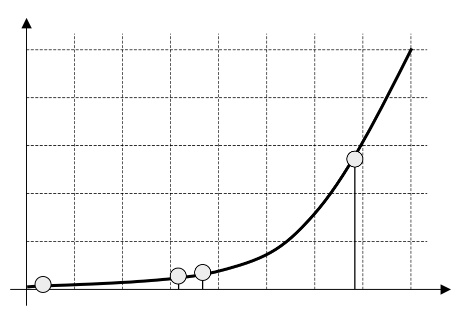
$$\Pr\left[\lim_{N\to\infty} \frac{1}{N}\sum_i^N \frac{f(x_i)}{p(x_i)} = I\right] = 1$$

MC Integration - Example

– Integral $I = \int_0^1 5x^4 dx = 1$

– Uniform sampling

– Samples :



x_1 = .86 <I> = 2.74

x_2 = .41 <I> = 1.44

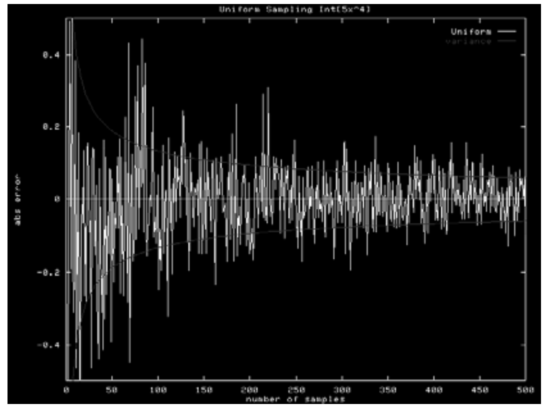
x_3 = .02 <I> = 0.96

x_4 = .38 <I> = 0.75

MC Integration - Example

– Integral

$$I = \int_0^1 5x^4 dx = 1$$



– Stochastic error

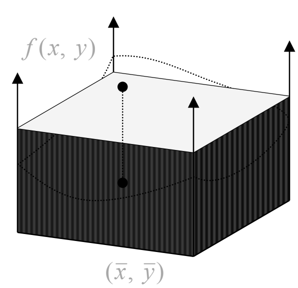
– Variance

- What is it?

MC Integration: 2D

- Primary estimator:

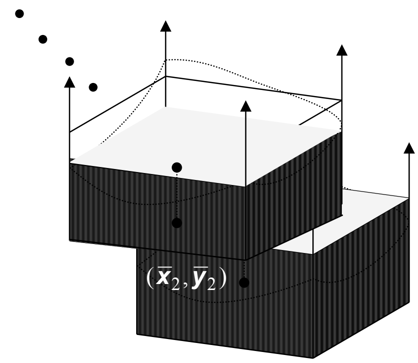
$$\bar{I}_{prim} = \frac{f(\bar{x}, \bar{y})}{p(\bar{x}, \bar{y})}$$



MC Integration: 2D

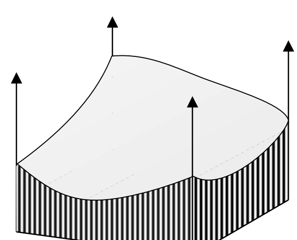
- Secondary estimator:

$$I_{\sec} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\bar{x}_i,\bar{y}_i)}{p(\bar{x}_i,\bar{y}_i)}$$



Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature



$$I = \int_a^b\int_c^d f(x,y)\,dx\,dy$$

$$\langle I\rangle = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i,y_i)}{p(x_i,y_i)}$$

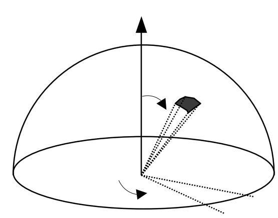
MC Advantages

- Convergence rate of O($\frac{1}{\sqrt{N}}$)
- Simple
  - Sampling
  - Point evaluation
  - Can use black boxes
- General
  - Works for high dimensions
  - Deals with discontinuities, crazy functions,…

MC Integration - 2D example

- Integration over hemisphere:

$$I = \int_{\Omega} f(\Theta) d\omega_{\Theta}$$
$$= \int_0^{2\pi} \int_0^{\pi/2} f(\varphi, \theta) \sin\theta \, d\theta \, d\varphi$$

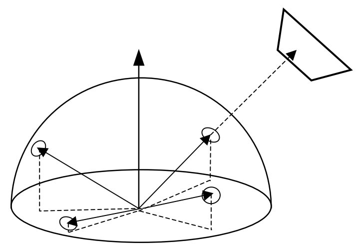


$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\varphi_i, \theta_i) \sin\theta}{p(\varphi_i, \theta_i)}$$

Hemisphere Integration example

Irradiance due to light source:

$$I = \int_{\Omega} L_{source} \cos\theta d\omega_{\Theta}$$
$$= \int_0^{2\pi}\int_0^{\pi/2} L_{source} \cos\theta \sin\theta d\theta d\varphi$$

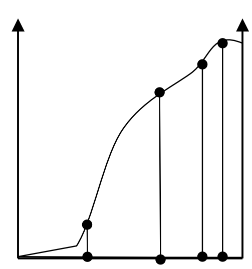


$$p(\omega_i) = \frac{\cos\theta \sin\theta}{\pi}$$

$$\langle I \rangle = \frac{1}{N}\sum_{i=1}^{N} \frac{L_{source}(\omega_i)\cos\theta\sin\theta}{p(\omega_i)} = \frac{\pi}{N}\sum_{i=1}^{N} L_{source}(\omega_i)$$

Importance Sampling

- Better use of samples by taking more samples in 'important' regions, i.e. where the function is large



MC integration - Non-Uniform

- Some parts of the integration domain have higher importance
- Generate samples according to density function p(x)

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

- Estimator?
- What is optimal p(x)? $p(x) \approx f(x) / \int f(x) dx$

MC integration - Non-Uniform

- Generate samples according to density function p(x)

$$p(x) \approx f(x) / \int f(x) dx$$

- Why? $I_{estimator} = \frac{1}{N} \sum \frac{f(x)}{p(x)} = \frac{1}{N} \sum \frac{f(x)}{f(x)/I} = \frac{1}{N} \sum I = I$

- But…..

$$\sigma^2 = \frac{1}{N} \int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x) dx$$

$$= \frac{1}{N} \int_a^b [\frac{f(x)}{f(x)/I} - I]^2 p(x) dx = 0$$

Example

- Function: $I = \int_0^4 x dx = 8$ $\quad f(x) = x$

$$\sigma^2 = \frac{1}{N}\int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x) dx$$

$$p(x) = \frac{x}{8}, \sigma^2 = 0 \qquad I_{estimator} = I = 8$$

$$p(x) = \frac{1}{4}, \sigma^2 = \frac{1}{N}\int_0^4 [\frac{x}{1/4} - 8]^2 \frac{1}{4} dx = 21.3/N$$

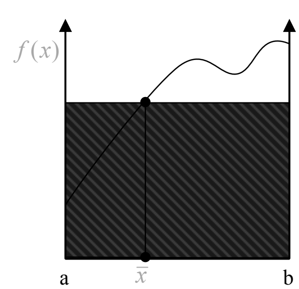
$$p(x) = \frac{x+2}{16}, \sigma^2 = \frac{1}{N}\int_0^4 [\frac{x}{(x+2)/16} - 8]^2 \frac{x+2}{16} dx = 6.3/N$$

Monte Carlo Integration

Primary estimator:

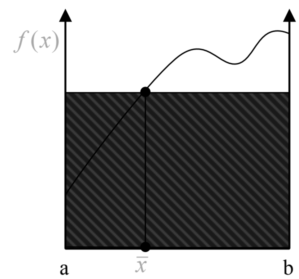
$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(\bar{x})(b-a)$$



Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})(b-a)$$

$$E(I_{prim}) = \int_a^b f(x)(b-a)p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator:

$$\sigma^2_{prim} = \int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

$\rightarrow$ a measure of the stochastic error

More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{\text{sec}} = \frac{1}{N}\sum_{i=1}^{N} f(\bar{x}_i)(b-a)$

Variance $\sigma_{\text{sec}}^2 = \sigma_{prim}^2 / N$