Lecture 5: Rendering Equation
Chapter 2 in Advanced GI

Fall 2004
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Radiometry

• Radiometry: measurement of light energy
• Defines relation between
  – Power
  – Energy
  – Radiance
  – Radiosity

Power

• Energy: Symbol: \( Q \); unit: Joules
• Power: Energy per unit time \( (dQ/dt) \)
  – Aka. “radiant flux” in this context
• Symbol: \( P \) or \( \Phi \); unit: Watts (Joules / sec)
  – Aka. “radiant flux” in this context
  – Photons per second
  – All further quantities are derivatives of \( P \)
    (flux densities)

Irradiance

• Power per unit area \( (dP/dA) \)
  – That is, area density of power
  – It is defined with respect to a surface
• Symbol: \( E \); unit: W / m²
  – Measurable as power on a small-area detector
  – Area power density exiting a surface is called radiant exitance \( (M) \)
    or radiosity \( (B) \) but has the same units

Radiance

• Radiance is radiant energy at \( x \) in direction \( \Theta \): 5D function
  – \( L(x \rightarrow \Theta) \) : Power
    • per unit projected surface area
    • per unit solid angle
    \[
    L(x \rightarrow \Theta) = \frac{d^2P}{dA^2d\Omega}
    \]
  – units: Watt / m².sr

Why is radiance important?

• Response of a sensor (camera, human eye) is proportional to radiance

• Pixel values in image proportional to radiance received from that direction
Relationships

- Radiance is the fundamental quantity
  \[ L(x \rightarrow \Theta) = \frac{d^2P}{dA'd\omega_\Theta} \]

- Power:
  \[ P = \int_{\text{area}} \int_{\text{angle}} L(x \rightarrow \Theta) \cdot \cos \Theta \cdot d\omega_\Theta \cdot dA \]

- Radiosity:
  \[ \dot{B} = \int_{\text{solid angle}} L(x \rightarrow \Theta) \cdot \cos \Theta \cdot d\omega_\Theta \]

Outline

- Light Model
- Radiance
- Materials: Interaction with light
- Rendering equation

Materials - Three Forms

- Ideal diffuse (Lambertian)
- Ideal specular
- Directional diffuse

BRDF

- Bidirectional Reflectance Distribution Function
  \[ f_s(x, \Psi \rightarrow \Theta) = \frac{dL(x \rightarrow \Theta)}{dE(x \leftarrow \Psi)} \]
  \[ = \frac{dL(x \rightarrow \Theta)}{L(x \leftarrow \Psi) \cos(N_x, \Psi) d\omega} \]

BRDF special case: ideal diffuse

Pure Lambertian
  \[ f_s(x, \Psi \rightarrow \Theta) = \frac{\rho_d}{\pi} \]

\[ \rho_d = \frac{\text{Energy}_{\text{out}}}{\text{Energy}_{\text{in}}} \quad 0 \leq \rho_d \leq 1 \]

Properties of the BRDF

- Reciprocity:
  \[ f_s(x, \Psi \rightarrow \Theta) = f_s(x, \Theta \rightarrow \Psi) \]

- Therefore, notation:
  \[ f_s(x, \Psi \leftrightarrow \Theta) \]

- Important for bidirectional tracing
Properties of the BRDF

- **Bounds:**
  \[ 0 \leq f_r(x, \Psi \leftrightarrow \Theta) \leq \infty \]

- **Energy conservation:**
  \[ \forall \Psi \int_0^{\infty} f_r(x, \Psi \leftrightarrow \Theta) \cos(N_x, \Theta) d\omega_0 \leq 1 \]

Outline

- **Light Model**

- **Radiance**

- **Materials: Interaction with light**

- **Rendering equation**

Light Transport

- **Goal**
  – Describe steady-state radiance distribution in scene

- **Assumptions:**
  – Geometric Optics
  – Achieves steady state instantaneously

- **Related:**
  – Neutron Transport (neutrons)
  – Gas Dynamics (molecules)

Radiance represents equilibrium

- **Radiance values at all points in the scene and in all directions expresses the equilibrium**
- **4D function: only on surfaces**

Rendering Equation (RE)

- **RE describes energy transport in scene**

- **Input**
  – Light sources
  – Surface geometry
  – Reflectance characteristics of surfaces

- **Output:** value of radiance at all surface points in all directions

Rendering Equation

\[ L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + L_r(x \rightarrow \Theta) \]
Rendering Equation

\[ L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\text{hemisphere}} L(x \leftarrow \Psi) f_r(x, \Psi \leftrightarrow \Theta) \cos(N_x, \Psi) d\omega_x \]

• Applicable for each wavelength

Summary

• Geometric Optics

• Goal:
  – to compute steady-state radiance values in scene

• Rendering equation:
  – mathematical formulation of problem that global illumination algorithms must solve
Shading Models

Ideal Specular Reflection
- Calculated from Fresnel’s equations
- Exact for polished surfaces
- Basis of early ray-tracing methods

Fresnel Equations
\[ \eta_1 \sin \theta_1 = \eta_2 \sin \theta_2 \]
\[ F_r = \frac{\eta_2 \cos \theta_1 - \eta_1 \cos \theta_2}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_2} \]
\[ F_i = \frac{\eta_1 \cos \theta_1 - \eta_2 \cos \theta_2}{\eta_1 \cos \theta_1 + \eta_2 \cos \theta_2} \]

Fresnel Reflectance
\[ F = \frac{(|F_r|^2 + |F_i|^2)}{2} \] for unpolarized light
- Equations apply for metals and nonmetals
  - for metals, use complex index \( \eta = n + ik \)
  - for nonmetals, \( k=0 \)

Metal vs. Nonmetal
Fresnel reflectance graph
Mies van der Rohe’s unbuilt Courtyard House

Ideal Diffuse Reflection
- Characteristic of multiple scattering materials
- An idealization but reasonable for matte surfaces
- Basis of most radiosity methods
- BRDF is a constant function

Directional Diffuse Reflection
- Characteristic of most rough surfaces
- Described by the BRDF

Classes of Models for the BRDF
- Plausible simple functions
  - Phong 1975;
- Physics-based models
  - Cook/Torrance, 1981; He et al. 1992;
- Empirically-based models
  - Ward 1992, Lafortune model

Phong Reflection Model
- Diffuse $= k_d (\vec{N} \cdot \vec{L})$
- Specular $= k_s (\vec{R} \cdot \vec{V})^n$
- $f_r(\Theta \leftrightarrow \Psi) = k_s (\frac{(R \cdot \Theta)^n}{(\vec{N}, \vec{V})}) + k_d$

The Blinn-Phong Model
- $f_r(\Theta \leftrightarrow \Psi) = k_s (\frac{(\vec{N}, \vec{H})^n}{(\vec{N}, \vec{V})}) + k_d$
**Phong: Reality Check**

**Phong model**

- Computationally simple, visually pleasing
- Doesn’t represent physical reality
  - Energy not conserved
  - Not reciprocal
  - Maximum always in specular direction

**Physics-based model**

**The Modified Blinn-Phong Model**

\[ f_s(\Theta \leftrightarrow \Psi) = k_s(N \cdot H)^s + k_d \]

**Cook-Torrance BRDF Model**

- A microfacet model
  - Surface modeled as random collection of planar facets
  - Incoming ray hits exactly one facet, at random
- Input: probability distribution of facet angle

**Facet Reflection**

- H vector used to define facets that contribute

**Cook-Torrance BRDF Model**

- “Specular” term (really directional diffuse)
- Fresnel reflectance for smooth facet

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Facet Distribution

- $D$ function describes distribution of $H$
- Formula due to Beckmann
  - derivation based on Gaussian height distribution

\[ D = \frac{1}{m^2 \cos^4 \alpha} e^{-\left(\frac{\tan \alpha}{m}\right)} \]

Masking and Shadowing

\[ G = \min \left[ 1, \frac{2(N \cdot H)(N \cdot V)}{(V \cdot H)^2}, \frac{2(N \cdot H)(N \cdot L)}{(V \cdot H)^2} \right] \]

Empirical BRDF Representation

- Generalized Phong model (Lafortune 1997)
- Used to represent:
  - Measured data
  - Wave optics reflectance model
- Features:
  - Efficient and compact
  - Easily added to rendering algorithms

Rob Cook’s vases

Source: Cook, Torrance 1981

Ward Model

- Physically valid
  - Energy conserving
  - Satisfies reciprocity: $f_r(\Theta_i \rightarrow \Theta_r) = f_r(\Theta_r \rightarrow \Theta_i)$
- Based on empirical data
- Isotropic and anisotropic materials

Ward Model: Isotropic

\[ f_r = \rho_r \frac{1}{4\pi \alpha^2} \exp\left(\frac{-\tan^2 \theta}{\alpha^2}\right) \frac{1}{\sqrt{(\vec{N} \cdot \vec{V})(\vec{N} \cdot \vec{L})}} \]

- where,
  - $\alpha$ is surface roughness
Ward Model: Anisotropic

\[ f_s = \rho_s \frac{1}{4\pi \alpha_x \alpha_y} \sqrt{\frac{N \cdot L}{N \cdot V}} \exp(-2 \frac{(\hat{H} \cdot \hat{x})^2 + (\hat{H} \cdot \hat{y})^2}{1 + \hat{H} \cdot \hat{N}}). \]

- where,
  - \( \alpha_x, \alpha_y \) are surface roughness in \( \hat{x}, \hat{y} \)
  - \( \hat{x}, \hat{y} \) are mutually perpendicular to the normal