Lecture 5: Rendering Equation
Chapter 2 in Advanced GI

Fall 2004

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Radiometry

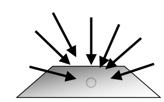
- Radiometry: measurement of light energy
- Defines relation between
  – Power
  – Energy
  – Radiance
  – Radiosity

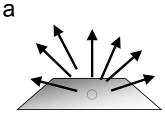
Power

- Energy: Symbol: *Q*; unit: Joules
- Power: Energy per unit time (*dQ*/*dt*)
  – Aka. "radiant flux" in this context
- Symbol: *P* or Φ; unit: Watts (Joules / sec)
  – Photons per second
  – All further quantities are derivatives of *P* (flux densities)

Irradiance

- Power per unit area (*dP*/*dA*)
  – That is, area density of power
  – It is defined with respect to a surface
- Symbol: *E*; unit: W / m^2
  – Measurable as power on a small-area detector
  – Area power density exiting a surface is called *radiant exitance* (*M*) or *radiosity* (*B*) but has the same units



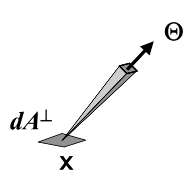


Radiance

- Radiance is radiant energy at x in direction θ: 5D function
  – $L(x \rightarrow \Theta)$: Power
    - per unit projected surface area
    - per unit solid angle

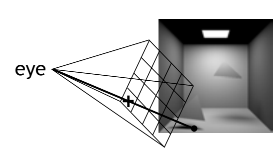
$$L(x \rightarrow \Theta) = \frac{d^2 P}{dA^{\perp} d\omega_{\Theta}}$$

  – units: Watt / m^2.sr



Why is radiance important?

- Response of a sensor (camera, human eye) is proportional to radiance



- Pixel values in image proportional to radiance received from that direction

Relationships

- Radiance is the fundamental quantity

$$L(x \to \Theta) = \frac{d^2 P}{dA^{\perp} d\omega_{\Theta}}$$

- Power:

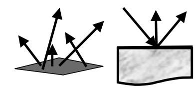
$$P = \int_{Area} \int_{Solid\ Angle} L(x \to \Theta) \cdot \cos\theta \cdot d\omega_{\Theta} \cdot dA$$

- Radiosity:

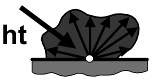
$$B = \int_{Solid\ Angle} L(x \to \Theta) \cdot \cos\theta \cdot d\omega_{\Theta}$$

Outline

- Light Model

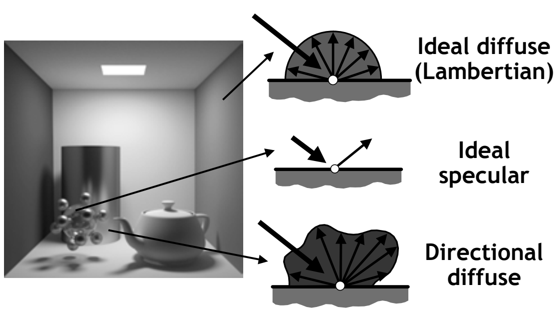


- Radiance
- **Materials: Interaction with light**



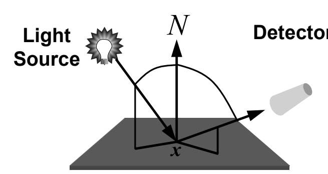
- Rendering equation

Materials - Three Forms



BRDF

- Bidirectional Reflectance Distribution Function

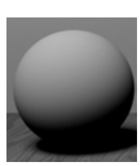


$$f_r(x, \Psi \to \Theta) = \frac{dL(x \to \Theta)}{dE(x \leftarrow \Psi)} = \frac{dL(x \to \Theta)}{L(x \leftarrow \Psi)\cos(N_x, \Psi) d\omega_{\Psi}}$$

BRDF special case: ideal diffuse

Pure Lambertian

$$f_r(x, \Psi \to \Theta) = \frac{\rho_d}{\pi}$$



$$\rho_d = \frac{Energy_{out}}{Energy_{in}} \qquad 0 \le \rho_d \le 1$$

Properties of the BRDF

- Reciprocity:

$$f_r(x, \Psi \to \Theta) = f_r(x, \Theta \to \Psi)$$

- Therefore, notation: $f_r(x, \Psi \leftrightarrow \Theta)$
- Important for bidirectional tracing

Properties of the BRDF

- Bounds:

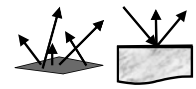
$$0 \leq f_r(x, \Psi \leftrightarrow \Theta) \leq \infty$$

- Energy conservation:

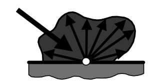
$$\forall \Psi \quad \int_{\Theta} f_r(x, \Psi \leftrightarrow \Theta) \cos(N_x, \Theta) d\omega_{\Theta} \leq 1$$

Outline

- Light Model



- Radiance
- Materials: Interaction with light



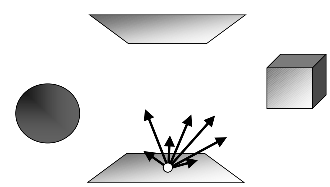
- Rendering equation

Light Transport

- Goal
  - Describe steady-state radiance distribution in scene
- Assumptions:
  - Geometric Optics
  - Achieves steady state instantaneously
- Related:
  - Neutron Transport (neutrons)
  - Gas Dynamics (molecules)

Radiance represents equilibrium

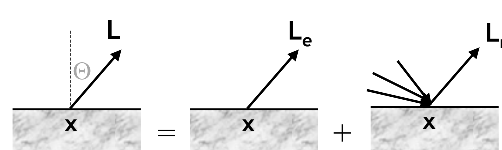
- Radiance values at all points in the scene and in all directions expresses the equilibrium
- 4D function: only on surfaces



Rendering Equation (RE)

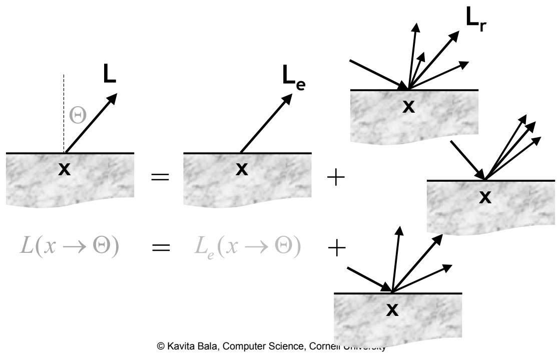
- RE describes energy transport in scene
- Input
  - Light sources
  - Surface geometry
  - Reflectance characteristics of surfaces
- Output: value of radiance at all surface points in all directions

Rendering Equation



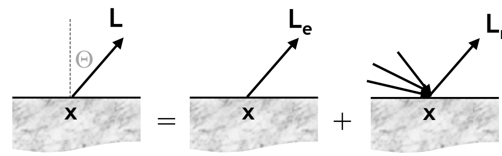
$$L(x \to \Theta) = L_e(x \to \Theta) + L_r(x \to \Theta)$$

Rendering Equation



$$L(x \to \Theta) = L_e(x \to \Theta) +$$

Rendering Equation



$$L(x \to \Theta) = L_e(x \to \Theta) + \int_{hemisphere} L(x \leftarrow \Psi) \cdots$$

Rendering Equation

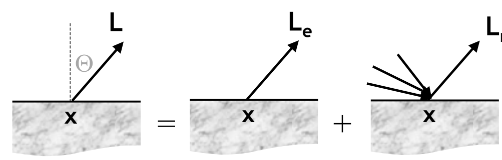
$$f_r(x, \Psi \leftrightarrow \Theta) = \frac{dL(x \to \Theta)}{dE(x \leftarrow \Psi)}$$

$$dL(x \to \Theta) = f_r(x, \Psi \leftrightarrow \Theta) dE(x \leftarrow \Psi)$$

$$dL(x \to \Theta) = f_r(x, \Psi \leftrightarrow \Theta) L(x \leftarrow \Psi) \cos(N_x, \Psi) d\omega_\Psi$$

$$L_r(x \to \Theta) = \int_{hemisphere} f_r(x, \Psi \leftrightarrow \Theta) L(x \leftarrow \Psi) \cos(N_x, \Psi) d\omega_\Psi$$

Rendering Equation

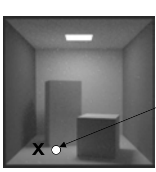


$$L(x \to \Theta) = L_e(x \to \Theta) +$$

$$\int_{hemisphere} L(x \leftarrow \Psi)\, f_r(x, \Psi \leftrightarrow \Theta) \cos(\mathbf{N}_x, \Psi) d\omega_\Psi$$

- **Applicable for each wavelength**

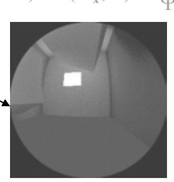
Rendering Equation



$$\underline{L(x \to \Theta)} = L_e(x \to \Theta) +$$

$$\int_{hemisphere} \underline{L(x \leftarrow \Psi)} f_r(x, \Psi \leftrightarrow \Theta) \cos(\mathbf{N}_x, \Psi) d\omega_\Psi$$

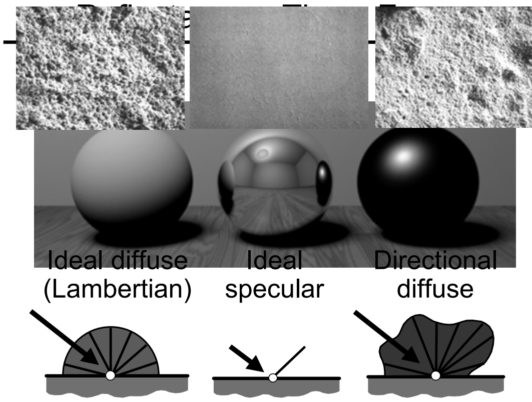
incoming radiance



Summary

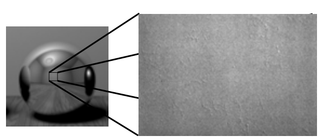
- Geometric Optics
- Goal:
  - to compute steady-state radiance values in scene
- Rendering equation:
  - mathematical formulation of problem that global illumination algorithms must solve

Shading Models

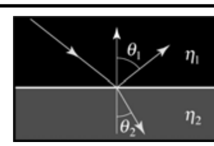


Ideal Specular Reflection

- Calculated from Fresnel's equations
- Exact for polished surfaces
- Basis of early ray-tracing methods



Fresnel Equations



$$\eta_1 \sin\theta_1 = \eta_2 \sin\theta_2$$

$$F_p = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

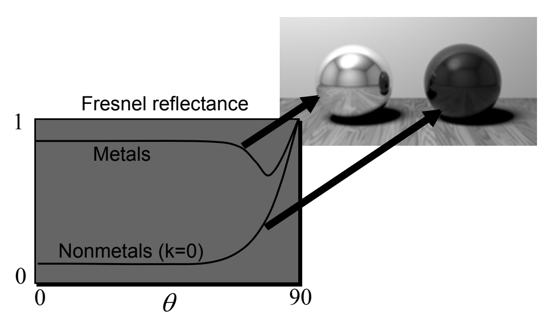
$$F_s = \frac{\eta_1 \cos\theta_1 - \eta_2 \cos\theta_2}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}$$

Fresnel Reflectance

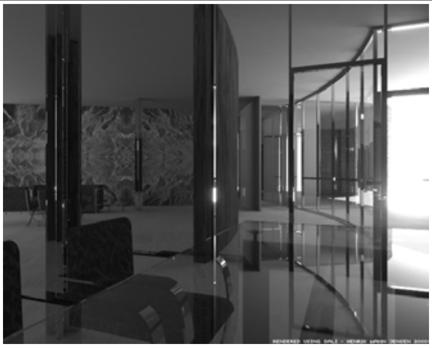
$$F = \frac{(|F_s|^2 + |F_p|^2)}{2}$$ for unpolarized light

- Equations apply for metals and nonmetals
  - for metals, use complex index $\eta = n+ik$
  - for nonmetals, $k=0$

Metal vs. Nonmetal

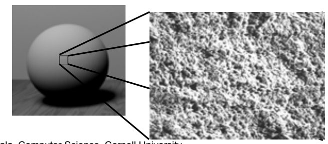


Mies van der Rohe's unbuilt Courtyard House



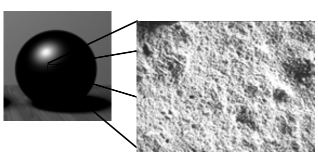
Ideal Diffuse Reflection

- Characteristic of multiple scattering materials
- An idealization but reasonable for matte surfaces
- Basis of most radiosity methods
- BRDF is a constant function



Directional Diffuse Reflection

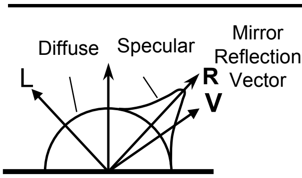
- Characteristic of most rough surfaces
- Described by the BRDF

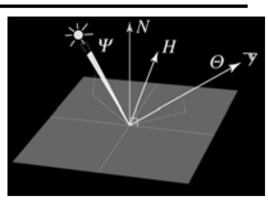


Classes of Models for the BRDF

- Plausible simple functions
  - Phong 1975;
- Physics-based models
  - Cook/Torrance, 1981; He et al. 1992;
- Empirically-based models
  - Ward 1992, Lafortune model

Phong Reflection Model

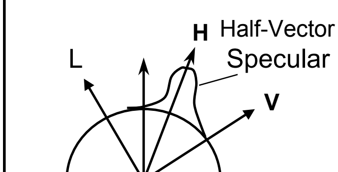


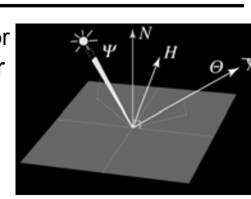


Diffuse = $k_d(\overline{N}\cdot\overline{L})$ Specular = $k_s(\overline{R}\cdot\overline{V})^n$

$$f_r(\Theta \leftrightarrow \Psi) = k_s \frac{(R.\Theta)^n}{(N.\Psi)} + k_d$$

The Blinn-Phong Model

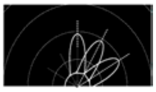




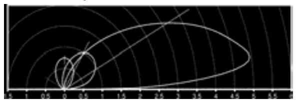
$$f_r(\Theta \leftrightarrow \Psi) = k_s \frac{(N.H)^n}{(N.\Psi)} + k_d$$

Phong: Reality Check

Phong model

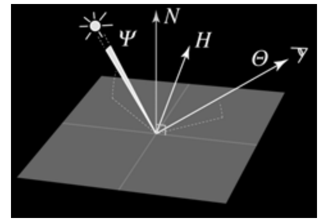


Physics-based model



- Computationally simple, visually pleasing
- Doesn't represent physical reality
  - – Energy not conserved
  - – Not reciprocal
  - – Maximum always in specular direction

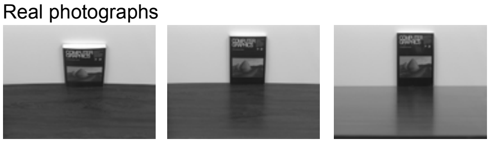
The Modified Blinn-Phong Model



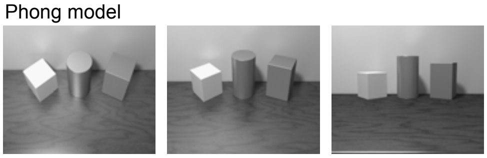
$$f_r(\Theta \leftrightarrow \Psi) = k_s (N.H)^n + k_d$$

Phong: Reality Check

Real photographs



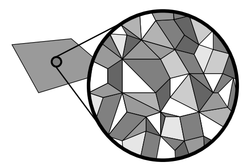
Phong model



Therefore, physically-based models

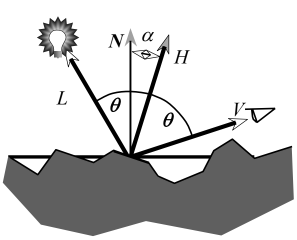
Cook-Torrance BRDF Model

- A *microfacet* model
  - – Surface modeled as random collection of planar facets
  - – Incoming ray hits exactly one facet, at random
- Input: probability distribution of facet angle



Facet Reflection

- *H* vector used to define facets that contribute



Cook-Torrance BRDF Model

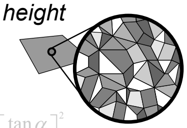
Fresnel Reflectance

$$R_s = \frac{F}{\pi} \frac{DG}{(N \cdot L)(N \cdot V)}$$

- "Specular" term (really directional diffuse)
- Fresnel reflectance for smooth facet

Facet Distribution

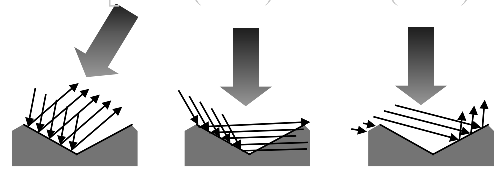
- D function describes distribution of H
- Formula due to Beckmann
  - *derivation based on Gaussian height distribution*



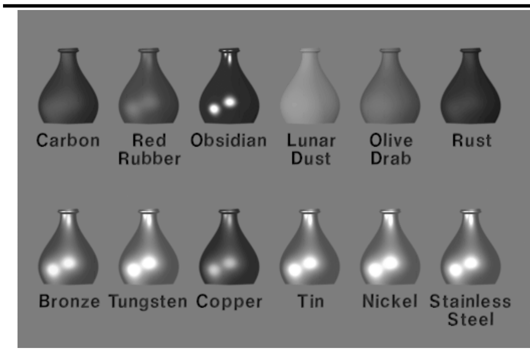
$$D = \frac{1}{m^2 \cos^4 \alpha} e^{-\left[\frac{\tan\alpha}{m}\right]^2}$$

Masking and Shadowing

$$G = \min\left[1, \frac{2(N \cdot H)(N \cdot V)}{(V \cdot H)}, \frac{2(N \cdot H)(N \cdot L)}{(V \cdot H)}\right]$$



Rob Cook's vases



Source: Cook, Torrance 1981

Empirical BRDF Representation

- Generalized Phong model (Lafortune 1997)
- Used to represent:
  - Measured data
  - Wave optics reflectance model
- Features:
  - Efficient and compact
  - Easily added to rendering algorithms

Ward Model

- Physically valid
  - Energy conserving
  - Satisfies reciprocity: $f_r(\Theta_i \to \Theta_r) = f_r(\Theta_r \to \Theta_i)$
- Based on empirical data
- Isotropic and anisotropic materials

Ward Model: Isotropic

$$f_s = \rho_s \frac{1}{4\pi\alpha^2} \frac{\exp(-\frac{\tan^2\theta_h}{\alpha^2})}{\sqrt{(\vec{N}\cdot\vec{V})(\vec{N}\cdot\vec{L})}}$$

- where,
  - α is surface roughness

Ward Model: Anisotropic

$$f_s = \rho_s \frac{1}{4\pi\alpha_x\alpha_y}\sqrt{\frac{\vec{N}\cdot\vec{L}}{\vec{N}\cdot\vec{V}}}\exp(-2\frac{\left(\frac{\vec{H}\cdot\hat{x}}{\alpha_x}\right)^2+\left(\frac{\vec{H}\cdot\hat{y}}{\alpha_y}\right)^2}{1+\vec{H}\cdot\vec{N}})$$

- where,
  - α_x, α_y are surface roughness in $\hat{x}, \hat{y}$
  - $\hat{x}, \hat{y}$ are mutually perpendicular to the normal