CS 664 Slides #9
Multi-Camera Geometry

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Pinhole Camera

- Geometric model of camera projection
  - Image plane I, which rays intersect
  - Camera center C, through which all rays pass
  - Focal length f, distance from I to C
Pinhole Camera Projection

- Point \((X,Y,Z)\) in space and image \((x,y)\) in \(I\)
  - Simplified case
    - \(C\) at origin in space
    - \(I\) perpendicular to \(Z\) axis

\[
x = \frac{fX}{Z} \quad \text{and} \quad y = \frac{fY}{Z}
\]
Homogeneous Coordinates

- Geometric intuition useful but not well suited to calculation
  - Projection not linear in Euclidean plane but is in projective plane (homogeneous coords)

- For a point \((x,y)\) in the plane
  - Homogeneous coordinates are \((\alpha x, \alpha y, \alpha)\) for any nonzero \(\alpha\) (generally use \(\alpha=1\))
    - Overall scaling unimportant
      \((X,Y,W) = (\alpha X, \alpha Y, \alpha W)\)
  - Convert back to Euclidean plane
    \((x,y) = (X/W, Y/W)\)
Lines in Homogeneous Coordinates

- Consider line in Euclidean plane
  \[ ax + by + c = 0 \]
- Equation unaffected by scaling so
  \[ aX + bY + cW = 0 \]
  \[ u^T p = p^T u = 0 \] (point on line test, dot product)
  - Where \( u = (a, b, c)^T \) is the line
  - And \( p = (X, Y, W)^T \) is a point on the line \( u \)
  - So points and lines have same representation in projective plane (i.e., in h.c.)
  - Parameters of line
    - Slope \( -a/b \), x-intercept \( -c/a \), y-intercept \( -c/b \)
Lines and Points

- Consider two lines
  \[ a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0 \]
  - Can calculate their intersection as
    \[ \left( \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \right) \]

- In homogeneous coordinates
  \[ u_1 = (a_1, b_1, c_1) \text{ and } u_2 = (a_2, b_2, c_2) \]
  - Simply cross product \( p = u_1 \times u_2 \)
    - Parallel lines yield point not in Euclidean plane

- Similarly given two points
  \[ p_1 = (X_1, Y_1, W_1) \text{ and } p_2 = (X_2, Y_2, W_2) \]
  - Line through the points is simply \( u = p_1 \times p_2 \)
Collinearity and Coincidence

- Three points collinear (lie on same line)
  - Line through first two is $p_1 \times p_2$
  - Third point lies on this line if $p_3^T(p_1 \times p_2) = 0$
  - Equivalently if $\det[p_1 \ p_2 \ p_3] = 0$

- Three lines coincident (intersect at one point)
  - Similarly $\det[u_1 \ u_2 \ u_3] = 0$
  - Note relation of determinant to cross product
    $u_1 \times u_2 = (b_1 c_2 - b_2 c_1, a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1)$

- Compare to geometric calculations
Back to Simplified Pinhole Camera

- Geometrically saw $x = \frac{fX}{Z}$, $y = \frac{fY}{Z}$

\[
\begin{pmatrix}
  fX \\
  fY \\
  Z
\end{pmatrix} = \begin{bmatrix}
  f & 0 & 0 \\
  0 & f & 0 \\
  1 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
  X \\
  Y \\
  Z
\end{pmatrix}
\]

3x4 Projection Matrix
Principal Point Calibration

- Intersection of principal axis with image plane often not at image origin

\[
\begin{pmatrix}
  fX + Zp_x \\
  fY + Zp_y \\
  Z
\end{pmatrix} = 
\begin{bmatrix}
  f & p_x & 0 \\
  f & p_y & 0 \\
  1 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
  X \\
  Y \\
  Z
\end{pmatrix}
\]

\( K = \begin{bmatrix}
  f & p_x \\
  f & p_y \\
  1 & 1
\end{bmatrix} \) (Intrinsic Calibration matrix)
CCD Camera Calibration

- Spacing of grid points
  - Effectively separate scale factors along each axis composing focal length and pixel spacing

\[
K = \begin{bmatrix}
m_x f & p_x \\
m_y f & p_y \\
1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\alpha & p_x \\
\beta & p_y \\
1 & 1
\end{bmatrix}
\]
Camera Rigid Motion

- Projection \( P = K[R|t] \)
  - Camera motion: alignment of 3D coordinate systems
  - Full extrinsic parameters beyond scope of this course, see “Multiple View Geometry” by Hartley and Zisserman
Two View Geometry

- Point X in world and two camera centers C, C’ define the epipolar plane
  - Images x,x’ of X in two image planes lie on this plane
  - Intersection of line CC’ with image planes define special points called epipoles, e,e’
Epipolar Lines

- Set of points that project to $x$ in $I$ define line $\ell'$ in $I'$
  - Called epipolar line
  - Goes through epipole $e'$
  - A point $x$ in $I$ thus maps to a point on $\ell'$ in $I'$
    - Rather than to a point anywhere in $I$
Epipolar Geometry

- Two-camera system defines one parameter family (pencil) of planes through baseline \(CC'\)
  - Each such plane defines matching epipolar lines in two image planes
  - One parameter family of lines through each epipole
  - Correspondence between images
Converging Stereo Cameras

Corresponding points lie on corresponding epipolar lines.

Known camera geometry so 1D not 2D search!
Motion Examples

- Epipoles in direction of motion

Parallel to Image Plane

Forward
Final Project

- Feel free to pick any vision related topic but discuss with me first
- Email choice to me by Tuesday, 11/18
  - Projects due Tuesday 12/16
- Suggested topics
  - Video insertion using affine motion estimation
  - Panoramic mosaics
  - Synthesis of novel views from stereo
  - Hausdorff based learning and matching
  - Flexible template matching
  - Stereo or motion using belief propagation
Fundamental and Essential Matrix

- Linear algebra formulation of the epipolar geometry
- Fundamental matrix, \( F \), maps point \( x \) in \( I \) to corresponding epipolar line \( \ell' \) in \( I' \)
  \[
  \ell' = Fx
  \]
  - Determined for particular camera geometry
    - For stereo cameras only changes if cameras move with respect to one another
- Essential matrix, \( E \), when camera calibration (intrinsic parameters) known
Fundamental Matrix

- Epipolar constraint
  \[ x'^T F x = x'^T l' = 0 \]
  - Thus from enough corresponding pairs of points in the two images can solve for \( F \)
    - However not as simple as least squares minimization because \( F \) not fully general matrix

- Consider form of \( F \) in more detail

\[ F = AL \]
Form of Fundamental Matrix

- **L**: \( x \rightarrow \ell 

- Epipolar line \( \ell \) goes through \( x \) and epipole \( e \)
- Epipole determines \( L \)
  \[ \ell = x \times e \]
  \[ \ell = Lx \quad \text{(rewriting cross product)} \]
- If \( e=(u,v,w) \)

\[
L = \begin{bmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{bmatrix}
\]

- \( L \) is rank 2 and has 2 d.o.f.
Form of Fundamental Matrix

- **A**: $\ell \rightarrow \ell'$
  - Constrained by 3 pairs of epipolar lines
    \[ \ell'_i = A \ell_i \]
  - Note only 5 d.o.f.
    - First two line correspondences each provide two constraints
    - Third provides only one constraint as lines must go through intersection of first two

- **F=AL** rank 2 matrix with 7 d.o.f.
  - As opposed to 8 d.o.f. in 3x3 homogeneous system
Properties of $F$

- Unique 3x3 rank 2 matrix satisfying $x'\mathbf{F}x=0$ for all pairs $x,x'$
  - Constrained minimization techniques can be used to solve for $F$ given point pairs
- $F$ has 7 d.o.f.
  - 3x3 homogeneous (9-1=8), rank 2 (8-1=7)
- Epipolar lines $\ell'=Fx$ and reverse map $\ell=F^Tx'$
  - Because also $(Fx)^T x'=0$ but then $x^T(F^Tx')=0$
- Epipoles $e'^T F=0$ and $Fe=0$
  - Because $e'^T \ell'=0$ for any $\ell'$; $Le=0$ by construction
Stereo (Epipolar) Rectification

- Given $F$, simplify stereo matching problem by warping images
  - Common image plane for two cameras
  - Epipolar lines parallel to $x$-axis
    - Epipole at $(1,0,0)$
    - Corresponding scan lines of two images
  - Intel vision library: calibration and rectification
Planar Rectification

- Move epipoles to infinity
  - Poor when epipoles near image
Stereo Matching

- Seek corresponding pixels in I, I’
  - Only along epipolar lines
- Rectified imaging geometry so just horizontal disparity D at each pixel
  \[ I'(x',y') = I(x+D(x,y),y) \]
- Best methods minimize energy based on matching (data) and discontinuity costs
Plane Homography

- Projective transformation mapping points in one plane to points in another
- In homogeneous coordinates

\[
\begin{pmatrix}
    aX+bY+cW \\
    dX+eY+fW \\
    gX+hY+iW
\end{pmatrix} =
\begin{bmatrix}
    a & b & c \\
    d & e & f \\
    g & h & i
\end{bmatrix}
\begin{pmatrix}
    X \\
    Y \\
    W
\end{pmatrix}
\]

- Maps four (coplanar) points to any four
  - Quadrilateral to quadrilateral
  - Does not preserve parallelism
Contrast with Affine

- Can represent in Euclidean plane $x' = Lx + t$
  - Arbitrary 2x2 matrix $L$ and 2-vector $t$
  - In homogeneous coordinates

$$\begin{pmatrix} aX + bY + cW \\ dX + eY + fW \\ W \end{pmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & i \end{bmatrix} \begin{pmatrix} X \\ Y \\ W \end{pmatrix}$$

- Maps three points to any three
  - Maps triangles to triangles
  - Preserves parallelism
Homography Example

- Changing viewpoint of single view
  - Correspondences in observed and desired views
  - E.g., from 45 degree to frontal view
    - Quadrilaterals to rectangles
  - Variable resolution and non-planar artifacts
Homography and Epipolar Geometry

- Plane in space $\pi$ induces homography $H$ between image planes

  $x' = H_\pi x$ for point $X$ on $\pi$, $x$ on $I$, $x'$ on $I'$
Obeys Epipolar Geometry

- Given $F, H_\pi$ no search for $x'$ (points on $\pi$)
  $$x'^T F x = 0, \quad x^T H_\pi^T F x = 0$$
- Maps epipoles, $e' = H_\pi e$
Computing Homography

- Correspondences of four points that are coplanar in world (no need for F)
  - Substantial error if not coplanar
- Fundamental matrix F and 3 point correspondences
  - Can think of pair e,e’ as providing fourth correspondence
- Fundamental matrix plus point and line correspondences
- Improvements
  - More correspondences and least squares
  - Correspondences farther apart
Plane Induced Parallax

- Determine homography of a plane
  - Remaining differences reflect depth from plane
Plane + Parallax Correspondences

\[ l' = x' \times Hx \]
Projective Depth

- Distance between $H_\pi x$ and $x'$ (along $l'$) proportional to distance of $X$ from plane $\pi$
  - Sign governs which side of plane
Multiple Cameras

- Similarly extensive geometry for three cameras
  - Known as tri-focal tensor
    - Beyond scope of this course

- Three lines
- Three points
- Line and 2 points
- Point and 2 lines