Jan 30, 2020

$$\min_x f(x) \quad f: \mathbb{R}^n \to \mathbb{R}, \quad f \text{ smooth}$$

Iterative: $x_{k+1} = G(x_k)$

step size

search direction

Idea: $x_{k+1} = x_k + \alpha_k p_k$

Taylor's theorem:

$$f(x_k + \varepsilon p_k) = \underbrace{f(x_k)}_{f_k} + \varepsilon p_k^T \underbrace{\nabla f(x_k)}_{g_k} + O(\varepsilon^2)$$

if $p_k^T g_k < 0$, then $f(x_k + \varepsilon p_k) < f(x_k)$ for small ε

$p_k = -g_k \quad p_k^T g_k = -g_k^T g_k = -\|g_k\|_2^2 < 0$

$$\boxed{\text{Gradient descent: } x_{k+1} = x_k - \alpha_k g_k}$$

$f(x) = \frac{1}{2} x^T A x + b^T x + c$, A SPD $\quad g(x) = Ax + b$

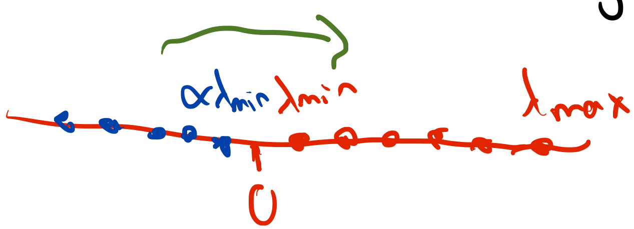
$$x_{k+1} = x_k - \alpha_k g_k \qquad x_{k+1} = x_k - \alpha(Ax_k + b)$$

at opt. $\quad x_* = x_* - \alpha(Ax_* + b)$

$$\overbrace{x_{k+1} - x^*}^{e_{k+1}} = \overbrace{x_k - x_*}^{e_k} - \alpha A(\overbrace{x_k - x_*}^{e_k})$$

$$e_{k+1} = (I - \alpha A)e_k \qquad \|e_{k+1}\|_2 \le \|I - \alpha A\|_2 \|e_k\|_2$$

$$\|I - \alpha A\|_2 = \max_j |1 - \alpha\lambda_j| = \max(|1-\alpha\lambda_{min}|, |1-\alpha\lambda_{max}|)$$



$$\alpha\lambda_{max} - 1 = 1 - \alpha\lambda_{min}$$

$$\alpha = \frac{2}{\lambda_{min} + \lambda_{max}}$$

$1 - \frac{2\lambda_{min}}{\lambda_{min} + \lambda_{max}}$

$$\|e_{k+1}\|_2 \le \|I - \alpha A\|_2 \|e_k\|_2$$

$\frac{\lambda_{min}}{\lambda_{max}}$ small $\Rightarrow$ slow convergence