April 18, 2019

Last time: network centrality

1. PageRank
   \[ \alpha P x + (1-\alpha) v = x \]

2. HITS (Hubs & Auth)
   \[ A = UEU^T \]
   \[ \text{auth} = U, \quad \text{hub} = U_1 \]

3. Eigenvector
   \[ A x = \lambda_1 x \]

4. Katz
   \[ (I - BA^T) R = BA^T \]
Hypergraphs
instances of small sets
- multiple recipients on email
- authors on a paper
- multiple drugs/medications for patients

Nodes

Hyperedges

Diagram: [hand-drawn diagram]
Adjacency tensor

simple model:
- size 3 relationships
- no direction

\[ A_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is in dataset} \\ 0 & \text{otherwise} \end{cases} \]

Tensor-based PageRank
(Gleich, Lim, Yu 2015)

\[ \alpha P x + (1-\alpha) v = x \]

What is "P" matrix for tensors?
columns normalized to sum to 1

"unfold"

normalize to sum to 1

refolding
Matrix case
\[ \alpha P x + (1 - \alpha) v = x \]

Tensor case
\[ \alpha P x^2 + (1 - \alpha) v = x \]

\[
\left[ P x^2 \right]_i = \sum_{j,k} P_{ijk} x_j x_k
\]

\[
\begin{pmatrix}
  x_1 x & x_2 x & \cdots & x_n x
\end{pmatrix}
\]

\[
= P x^2
\]

Does a solution even exist?
Yes!
Claim: \( f(x) = \alpha \sum x^2 + (1-\alpha)v \)
is a stochastic vector
\((1^T f(x) = 1 \quad f(x) \geq 0)\)

Proof:
\[
\sum x^2 = \begin{pmatrix} x_1 x & x_2 x & \cdots & x_n x \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}
\]

\( \sum_{i,j,k} \) is stochastic

\( y_j = \sum_{i,j} x_i y_i \) is stochastic

\( \sum x^2 = \sum x_j y_j \) is stochastic

\( f(x) = \alpha \sum x^2 + (1-\alpha)v \) is stochastic
\[ \alpha \frac{1}{2} x^2 + (1 - \alpha) \mathbb{1} = x \]

\( f(x) \) is stochastic or long as \( x \) is stochastic.

**Theorem (Brouwer fixed point):**

Let \( g : K \rightarrow K \) be a smooth function on a compact convex set \( K \). Then there exists \( x \in K \) such that \( g(x) = x \).

**Proof:** For \( K = [0, 1] \)

\[ h(x) = g(x) - x \]

\( h(0) \neq 0 \) \( \Rightarrow \) \( g(0) = y \)
\( K = \{ w \in \mathbb{R}^n \mid \sum w = 1, w \geq 0 \} \)

\( f : K \to K \)

\( f(x) = \alpha \frac{P}{2} x^2 + (1 - \alpha) v \)

\( \text{BFP} \Rightarrow \exists x \text{ such that} \)

\( \alpha \frac{P}{2} x^2 + (1 - \alpha) v = x \)

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Can we actually compute a solution?

Matrix

\( x_{k+1} = \alpha \frac{P}{2} x_k + (1 - \alpha) v \)

Tensor

\( x_{k+1} = f(x_k) = \alpha \frac{P}{2} x_k^2 + (1 - \alpha) v \)
Theorem (Gleich, Lim, Yu)

if $\alpha < \frac{1}{2}$

- * converges (again, exponential)

- solution $\alpha P_1 x^2 + (1-\alpha) v = x$

  is unique

if $\alpha \geq \frac{1}{2}$

- * doesn't necessarily converge

- solution may not be unique
Hypergraph eigenvector centrality
(Benson 2019)

Graph eigenvector centrality

\[ Ax = \lambda_1 x \quad x > 0 \]

\[ x_i = \gamma \sum_{j : (i, j) \in E} x_j \]

\[ \lambda_1 = \frac{1}{\gamma} \]

Theoretical tool:
Perron-Frobenius theorem

power of node i
prop. to powers of all such j, k
$x_i = \gamma \sum_{(i;j;k)} x_j x_k$

$\Rightarrow A x^2 = \frac{1}{\gamma} x$

2. $x_i^2 = \gamma \sum_{(i;j;k)} x_j x_k$

$\Rightarrow A x^2 = \frac{1}{\gamma} x^2$

$[x^2]_i = x_i$

H-eigenvector of $A$
\[ x_i = \sum_{(i,j,k)} x_j + x_k \]

\[ \Rightarrow Wx = \frac{1}{2\pi} x \]

\[ W_{ij} = \# \text{ of } k \text{ such that } A_{ijk} = 1 \]

\[ W = \sum_{k} A_{ijk} \]

Can apply P-F like last time

\[ \Rightarrow \text{unique } x > 0 \text{ such that } Wx = \lambda_1 x \]

\[ \lambda_1 > 0 \]
\( A x^2 = \lambda x \) { also want } \( A x^2 = \lambda x^2 \) \( x > 0 \)

Start with H

Definition: \( A \) is irreducible if \( W \) (\( W = \Sigma A_{i,j} \)) induces a connected graph

Theorem: If \( A \) is irreducible and nonnegative, then exists unique \( \lambda > 0, x > 0 \) such that

\( A x^2 = \lambda x^2 \)
any other eigenvalue $\lambda'$ has $|\lambda'| \leq 1$

$\Rightarrow$ this gives us the centrality vector

Can we compute it? Yes! (Ng, Qi, Zhou 2009)

\[ y_k = A^2 x_k \]

\[ x_{k+1} = \frac{y_k}{\sqrt{\|y_k\|_2}} \]

entry-wise $\sqrt{}$
2-eigenvector centrality

Want:

\[ A x^2 = \lambda x \]

\[ \lambda > 0, \ x > 0 \]

Theorem: if \( A \) is irreducible, then there exists \( \lambda > 0, x > 0 \) such that \( A x^2 = \lambda x \)

but...

- NP-hard to compute
- could be another \( \lambda' > 0, y > 0 \) such that \( A y^2 = \lambda' y \)
if we can compute some $\lambda > 0$, $x > 0$ such that

\[ A x^2 = \lambda x \]

then we have tensor $Z$-eigenvector centrality vector

Options:

- worst-case exponential time algorithms
- heuristics that tend to work very well in practice