1. Consider the Lax-Friedrichs method for a scalar conservation law \( u_t = -(f(u))_x \). Show that Lax-Friedrichs is actually second-order for a modified PDE \( u_t = -(f(u))_x + T \), where \( T \) is a sum of some other terms that you must determine, all of which should be \( O(k) \). The formula for \( T \) will involve \( h \) and \( k \). In your analysis, assume \( k/h = \lambda \), where \( \lambda \) is a fixed constant.

If you carry this out correctly, \( T \) will have three terms, two of which are multiples \( u_{xx} \).

If the two latter terms are added to arrive at a single term \( Au_{xx} \), is the coefficient \( A \) positive or negative? Take into account the CFL condition.

[Hint: Plug in the exact solution and carry out a Taylor series expansion. Figure out how to define \( u_t \) (i.e., figure out what \( T \) has to be) in such a way that the \( ku_t \) term on the left-hand side of the Taylor expansion can cancel the \( O(k^2) \) term on the left-hand side and the \( O(h^2) \) term on the right-hand side.]
4. Implement (in Matlab) the Lax-Friedrichs and Godunov method for Burgers’ equation with Riemann initial condition

\[ u_0(x) = \begin{cases} 
2 & \text{for } x < 0, \\
1 & \text{for } x \geq 0. 
\end{cases} \]

Use spatial interval \([-1, 1]\) and integrate up to \(T = 0.2\). Try various values of \(k\) and \(h\), ensuring that the CFL condition is satisfied. Use the obvious Dirichlet boundary conditions (2 at the left, 1 at the right).

Develop a heuristic to measure the width of the shock. Determine experimentally how much Lax-Friedrichs smears the shock (as a function of \(k\)), and how much Godunov smears the shock.

Hand in listings of all m-files, at least two interesting plots, and a paragraph of conclusions.