CS 624: Numerical Solution of Differential Equations
Spring 2002
**Problem Set 5**

Handed out: Wed., Apr. 10.

Due: Fri., Apr. 19 in lecture.

1. Show that following conservation-form method for the inviscid Burgers' equation is consistent and monotone.

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}(g(v_j^n, v_{j+1}^n) - g(v_{j-1}^n, v_j^n)),$$

where $g(a,b) = \max(a,0)^2 + \min(b,0)^2$. You can impose a CFL condition. (Read about the definition of monotone on pp. 169-170 of LeVeque.)

2. (a) Show that the viscous Burgers equation $u_t + uu_x = au_{xx}$, where $a > 0$ is a constant, can be transformed analytically to a diffusion equation.

[Hint: Use the "Cole-Hopf transformation": Show that if $v$ satisfies $v_t + v_x^2/2 = av_{xx}$, then $u = v_x$ is a Burgers solution. Then substitute $v = -2a \log w$, coming up with a new equation for $w$.]

(b) Note that the diffusion equation $w_t = aw_{xx}$ defined on $[0, 2\pi] \times [0, \infty)$ with periodic BC's (i.e., $w(0,t) = w(2\pi, t)$ for all $t \geq 0$) has as an exact analytic solution $w(x,t) = C + \exp(-n^2 at)\sin(nx)$ for any integer $n$ and real constant $C$. Figure out the corresponding Burgers' analytic solution.

3. Write down a semi-discretization of viscous Burgers' equation, that is, discretize in space only yielding a system of ODE's. Your discretization should be "consistent" in the intuitive sense that it is made up of terms that correspond to finite difference approximations to the terms in Burgers equation, but you do not need to formally define or prove that it is consistent. Your system of ODE's should be based on spatial domain $[0, 2\pi]$ including periodic boundary conditions.

4. Implement (in Matlab) the numerical method for the viscous Burgers equation in the last question. Use periodic boundary conditions ($u(0,t) = u(2\pi, t)$ for all $t \geq 0$). Use examples from 2(b) for initial conditions. As usual, you can use ODE15s to solve the ODEs.

Now compare (experimentally) your computed solution at time $t = 1$ to the exact analytic solution and try to determine (experimentally) the order of your method (order with respect to $h$). This may require some adjusting of the tolerances to ODE15s so that the role of $k$ is not an issue.

Hand in listings of all m-files, at least two interesting plots, and a paragraph of conclusions.