CS 624: Numerical Solution of Differential Equations
Spring 2002
Problem Set 3

Handed out: Wed., Feb. 27.

Due: Fri., Mar. 8 in lecture.

1. Consider the Kepler problem (light body orbiting heavy body, PS1 Q4). Let $H(\mathbf{x}, \mathbf{v})$ be the energy function used in PS1.

   (a) Show that if the leapfrog method is used to integrate the Kepler system, then $H(\mathbf{x}^{n+1}, \mathbf{v}^{n+1}) - H(\mathbf{x}^n, \mathbf{v}^n) = O(k^2)$. Explicitly find the coefficient (which should depend on $\mathbf{x}^n$ and $\mathbf{v}^n$) multiplying the k^2 term in the remainder.

   (b) Construct a perturbed energy function $\bar{H}(\mathbf{x}, \mathbf{v})$ (that may also depend on k) with the property that if the sequence $(\mathbf{x}^n, \mathbf{v}^n)$ is defined as above (i.e., as the solution computed by leapfrog to the original Kepler problem), then $\bar{H}(\mathbf{x}^{n+1}, \mathbf{v}^{n+1}) - \bar{H}(\mathbf{x}^n, \mathbf{v}^n) = O(k^3)$. Your function $\bar{H}(\mathbf{x}, \mathbf{v})$ should have the form

   $$\bar{H}(\mathbf{x}, \mathbf{v}) = H(\mathbf{x}, \mathbf{v}) + kf(\mathbf{x}, \mathbf{v}),$$

   where $f(\mathbf{x}, \mathbf{v})$ is a function that you determine explicitly. [Note that any algebraic mistake in your solution to part (a) will make part (b) impossible, so be careful with your algebra. This problem can be solved without any outside reading, but if you get stuck on (b), you will find a related solved example in the Sanz-Serna & Calvo book on reserve, pp. 129–131.]

2. Consider a partitioned Hamiltonian system of ODE's in the case $d = 1$ (i.e., the Hamiltonian has the form $H(p, q) = f(p) + g(q)$ where p, q are scalars). Can Euler's method ever be symplectic for such a problem? Determine conditions on $H(p, q)$ that are necessary and sufficient for EM to be symplectic.

3. Consider the upwind method $v_j^{n+1} = v_j^n + \lambda(v_{j+1}^n - v_j^n)$ for the first-order wave equation $u_t = u_x$, which is a first-order method. Show that this method is actually second-order for a *different* PDE, namely, for a certain advection-diffusion equation $u_t = u_x + au_{xx}$, where a may depend on k and/or h.

4. Symplectic integrators for Hamiltonian problems do not conserve energy, as mentioned in lecture. One way to get energy conservation for Hamiltonian problems is to drop one of the equations in the ODE system, and replace it with a constraint that energy is conserved. Then the resulting problem can be solved as a DAE. This method generally does not work very well.

   Solve the Kepler problem (PS1, Q4) in this manner. Use `ode15s`. Note that `ode15s` solves index-one DAE's in semi-explicit form if you give it a diagonal matrix M as input that has a mixture of 1's (evolution equations) and 0's (constraints) on the main diagonal. Try replacing one of the x_i evolution equations with the constraint, and also one of the v_i evolution equations. Hand in m-file listings, plots, and a brief statement of your conclusions. Explain why this method doesn't work very well. (Hint: consider the index.)