CS 624: Numerical Solution of Differential Equations
Spring 2002
Problem Set 1

Handed out: Wed., Jan. 30.

Due: Fri., Feb. 8 in lecture.

1. Exercise 1.3.1 of the text. In addition, consider the D-stability of the method using, e.g., the Matlab `roots` function.

2. The analysis of the order of the Backward Euler method presented in lecture did not exactly correspond to the analysis for the general case of LMS methods (both presented on 1/25/02), since the former was Taylor-expanded about t_{n+1} whereas the latter was expanded about t_n. Does this make a difference? Redo the analysis of the general LMS case with all expansions about t_{n+r} for a fixed constant r, not necessarily 0, and determine whether the order or leading error coefficient will be different from the $r = 0$ case done in lecture.

3. Let
$$(x_1, y_1), \ldots, (x_n, y_n), (w_1, z_1), \ldots, (w_m, z_m)$$
be a sequence of $n + m$ real points in the plane such that $x_1 < x_2 < \cdots < x_n < w_1 < \cdots < w_m$. Assume $n > 0$. Show that there exists a unique polynomial p of degree at most $n + m - 1$ such that $p(x_i) = y_i$ for $i = 1, \ldots, n$ and $p'(w_i) = z_i$ for $i = 1, \ldots, m$.

   [Hint: As in lecture, first show uniqueness, then call upon linear algebra to conclude existence. To show uniqueness, first argue that p' is uniquely determined. Rolle's theorem might be helpful. Note: this theorem is used to establish the validity of the interpolation-based definition of the BDF family.]

4. Consider a light body orbiting a heavy body located at the origin lying in a plane. The equation of motion of the light body is given by
$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{\mathbf{x}}{\|\mathbf{x}\|^3}.$$
(The norm in the denominator is the 2-norm.) Convert this to a first-order system. (You should end up with a total of four dependent variables.) Write AB1 and AB2 algorithms in Matlab and apply them to this problem. Set up initial conditions in which the light body starts at $(1, 1)$ and is moving with velocity $(0, 1)$. Hand in plots of the trajectories of the bodies for the same initial condition for both AB1 and AB2, using two or three different time-step choices. Note: initialize AB2 with a single step of AB1.

Energy for this problem is given by

$$H(\mathbf{x}, \mathbf{v}) = \frac{\|\mathbf{v}\|^2}{2} - \frac{1}{\|\mathbf{x}\|}$$

where $\mathbf{v} = d\mathbf{x}/dt$. It is easy to check using algebra that dH/dt is identically 0. Determine computationally how well the AB1 and AB2 methods conserve energy by plotting $H(\mathbf{x}(t), \mathbf{v}(t)) - H(\mathbf{x}(0), \mathbf{v}(0))$ as a function of t for both methods and for different time steps.

Turn in listings of your m-files, a paragraph or two of conclusions and at the requested plots.