CS 624: Numerical Solution of Differential Equations
Spring 2002
Prelim 2

Handed out: Tues., Mar. 26.

This exam has four questions. The questions are weighted equally. It counts for 20% of your final course grade (same as Prelim 1). This exam is due back at the end of lecture Friday, March 29 (if you picked up the exam on Tuesday, March 26) or at the end of lecture on Monday, April 1 (if you picked up the exam on Friday, March 29).

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Trefethen then you must cite them.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until Tuesday, April 2. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

You are not allowed to send any email or otherwise make any on-line posting concerning the questions on this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in after lecture but before 5:00 p.m. on the due date will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are handed in on time.

1. Consider applying the leapfrog method to the partitioned Hamiltonian problem whose Hamiltonian is given by $H(\mathbf{p}, \mathbf{q}) = (\mathbf{p}^T A\mathbf{p} + \mathbf{q}^T B\mathbf{q})/2$ where A, B are positive-definite diagonal matrices. Determine the condition that assures that the numerical method is stable (i.e., does not exhibit unbounded growth as $n \to \infty$). Your condition should be in as simple form as possible, involving k and the diagonal entries of A, B.

2. Transform the Kepler system of ODEs into polar coordinates. In particular, let the change of variables be $\mathbf{x} = r(\cos\theta, \sin\theta)$ and $\mathbf{v} = w(\cos\beta, \sin\beta)$ where r, θ, w, β are the new time-dependent variables. (Simplify as much as possible.) Argue that it is possible to replace one evolution equation in the transformed system by an energy conservation constraint in such a way that the resulting DAE system stays index-one throughout its trajectory.

3. Develop an explicit, one-step, stable, consistent method for the one-way wave equation in which v_j^{n+1} depends only on v_j^n and v_{j+2}^n. Verify stability and determine the order of your method.

4. Consider the following finite difference method for the full wave equation

$$v_j^{n+1} = 2v_j^n - v_j^{n-1} + \lambda^2(v_{j+1}^n - 2v_j^n + v_{j-1}^n),$$

which is due to Courant et al. Determine its order of accuracy, assuming $\lambda = \text{const}$.

[Hints: The full wave equation $u_{tt} = u_{xx}$ has second derivatives in time. On the other hand, the definition of "order of accuracy" in the text allows only first derivatives. Reduce the full wave equation to a system which has only first time derivatives by using the usual trick of introducing a velocity variable. Then try to rewrite the finite difference method above as a one-step method that computes a 2-vector, one entry of which is an approximation to u and another which is an approximation to u_t. Finally, determine the order using (4.2.4). A better analysis may result if the approximation to u is defined on time grid offset by $k/2$ from the time grid for u_t.]