CS 624: Numerical Solution of Differential Equations
Spring 2002
Practice Prelim 1

Handed out: Wed., Feb. 13.

This exam had five questions for a total of 75 points. The class had 75 minutes to answer all questions. No books or notes were allowed.

1. **[10 points]** Consider the one-step LMS formula $v^{n+1} = v^n + k(\beta_0 f^n + \beta_1 f^{n+1})$. What are conditions on β_0, β_1 that ensure that this method is first order? Second order? For your information, the definition of C_l that determines order is

$$C_l = \sum_{j=0}^{s} j^l \alpha_j - l \sum_{j=0}^{s} j^{l-1} \beta_j$$

   where the second term is absent when $l = 0$.

2. **[10 points]** Consider integrating an ODE with a fixed time step k, and alternating between a second and third order LMS rules (both D-stable). (In other words, v^n for n odd is computed with the second-order rule and for n even with the third-order rule.) Assume the initial conditions are exact. What form would you expect for the error $u(1) - v^{1/k}$, in terms of k? Explain.

3. **[15 points]** Consider the IVP $du/dt = au$, $u(0) = u_0$ where a is a negative real number and u_0 is a positive real number. Then the true solution is positive for all $t \geq 0$. It is sometimes desirable for the computed solution to also have that property. For both the Euler and Backward Euler methods, determine conditions on the positive time step k to guarantee that all v^n are positive.

4. **[20 points]** Returning to the setup of question 1, suppose $\beta_0 + \beta_1 = 1$, $\beta_0 \geq 0$, and $\beta_1 \geq 0$. Determine additional conditions for β_0, β_1 to ensure that the entire negative real axis is contained in the stability region of the method.

5. **[20 points]** Let f^n denote $f(v^n, t_n)$ as in lecture, and let g^n denote $f_x(v^n, t_n)$, that is, the derivative of f with respect to its first argument evaluated at (v^n, t_n). Analogously, let h^n denote $f_y(v^n, t_n)$. From the equation

$$u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} f(u(t), t)\, dt.$$

   derive an explicit one-step integration rule that uses f^n, g^n, h^n. In more detail, approximate the left-hand side with $v^{n+1} - v^n$, and approximate the right-hand side using a linear polynomial that interpolates $f(u(t), t)$ and its t-derivative at t_n.