Handed out: Fri., Apr. 7.

Due: Mon., Apr. 17 in lecture.

1. What is the order of accuracy of the Lax-Friedrichs method when applied to the one-way wave equation?

2. (a) Show that the viscous Burgers equation $u_t + uu_x = au_{xx}$, where $a > 0$ is a constant, can be transformed analytically to a diffusion equation.

   [Hint: Use the “Cole-Hopf transformation”: Show that if $v$ satisfies $v_t + v_x^2/2 = av_{xx}$, then $u = v_x$ is a Burgers solution. Then substitute $v = -2a \log w$, coming up with a new equation for $w$.]

   (b) Note that the diffusion equation $w_t = aw_{xx}$ defined on $[0, 2\pi] \times [0, \infty)$ with periodic BC’s (i.e., $w(0, t) = w(2\pi, t)$ for all $t \geq 0$) has as an exact analytic solution $w(x, t) = C + \exp(-n^2at)\sin(nx)$ for any integer $n$ and real constant $C$. Figure out the corresponding Burgers’ analytic solution.

3. Design an explicit numerical method for the viscous Burgers equation. Your method should be “consistent” in the intuitive sense that it is made up of terms that correspond to finite difference approximations to the three terms in Burgers equation, but you do not need to formally define or prove that it is consistent. Show that it is $l^1$-contracting for correct choices of $k$ and $h$. [Hint: try to mimic the proof from lecture that LF is $l^1$-contracting. Solutions to this problem will likely require small time steps, e.g., $k = O(h^2)$.]

4. Implement (in Matlab) the numerical method for the viscous Burgers equation in the last question. Use periodic boundary conditions ($u(0, t) = u(2\pi, t)$ for all $t \geq 0$). Use examples from 2(b) for initial conditions.

   Now compare (experimentally) your computed solution at time $t = 1$ to the exact analytic solution and determine (experimentally) the order of your method.

   Hand in listings of all m-files, at least two interesting plots, and a paragraph of conclusions.