Handed out: Fri., Feb. 4.

Due: Mon., Feb. 14 in lecture.

1. Exercise 1.3.1 of the text.

2. The analysis in lecture that led from the “motivational” version of the definition of order of an LMS formula to the “computational” version (i.e., the proof that involved expanding \( v^{n+s} - u((n+s)k) \) as a Taylor series in \( k \) had a flaw in it in the case of an implicit method. The flaw is that the analysis assumed \( f^{n+s} \) is exactly \( f(u((n+s)k), t_{n+s}) \). This assumption is invalid since the hypothesis of the proof is that \( u(0) = v_0, \ldots, u((n + s - 1)k) = v^{n+s-1} \), but \( u((n+s)k) \neq v^{n+s} \). Redo this analysis without making this invalid assumption. Your analysis should lead to the same conclusion about how to define “order” and should also yield the same value for the leading error coefficient as presented in lecture.

[Hint: Let \( d \) stand for \( v^{n+s} - u((n+s)k) \). By calling upon the mean value theorem, you can write \( f^{n+s} \) in terms of \( f(u((n+s)k), t_{n+s}) \) and \( d \). Make any necessary assumptions about differentiability. Then you get the same formula in class, except \( d \) appears not only on the left-hand side of the equation but also in a new term on the right-hand side. Solve for \( d \), and re-expand the right-hand side as a Taylor series in \( k \).]

[Note: The corresponding argument in the book on pp. 22-23 has the same flaw.]

3. Let
\[
(x_1, y_1), \ldots, (x_n, y_n), (w_1, z_1), \ldots, (w_m, z_m)
\]
be a sequence of \( n + m \) real points in the plane such that \( x_1 < x_2 < \cdots < x_n < w_1 < \cdots < w_m \). Assume \( n > 0 \). Show that there exists a unique polynomial \( p \) of degree \( n + m - 1 \) such that \( p(x_i) = y_i \) for \( i = 1, \ldots, n \) and \( p'(w_i) = z_i \) for \( i = 1, \ldots, m \).

[Hint: As in lecture, first show uniqueness, then call upon linear algebra to conclude existence. To show uniqueness, first argue that \( p' \) is uniquely determined. Rolle’s theorem might be helpful. Note: this theorem is used to establish the validity of the interpolation-based definition of the BDF family.]

4. Consider a frictionless pendulum, whose equations of motion are:
\[
\begin{align*}
\frac{d\theta}{dt} &= v, \\
\frac{dv}{dt} &= -\sin \theta
\end{align*}
\]
where $\theta$ is the angle made by the pendulum with respect to vertical and $v$ is the angular velocity of the pendulum. It can be verified mathematically that this system conserves energy, where energy is $-\cos \theta + v^2/2$.

Choose starting data $\theta_0 = \pi/2$ and $v_0 = 0$ (i.e., the pendulum is stationary and horizontal). Implement AB1 and AB2 for this system in Matlab and track the energy of the system. How well do each of them conserve energy for a fixed time step and fixed interval of integration? What happens to each when the time step is halved (but the interval of integration is fixed)?

Run AB1 for a very large number of steps. You will notice there is eventually a qualitative transition to a different kind of behavior. Can you explain this transition? (The same thing will happen to AB2, if the number of steps is large enough.)

Turn in listings of your m-files, a paragraph or two of conclusions and at least one interesting plot.