CS 624: Numerical Solution of Differential Equations  
Spring 2000  
Final Exam

Handed out: Fri., May 5.

This exam has four questions. The questions are weighted equally. It counts for 30% of your final course grade. This exam is due back at 12:05 three days after you picked it up (i.e., it is due either on Monday, May 8 or Thursday, May 11).

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than the textbooks, then you must cite them.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until 5 pm, Saturday, May 13 because a student is taking the exam late. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else’s lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: “I have neither given nor received unpermitted assistance on this exam.”

You are not allowed to send any email or otherwise make any on-line posting concerning the questions on this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in after 12:05 but before 5:00 p.m. on the due date will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are handed in on time.

1. Propose a second-order, one-step explicit method for the one-way wave equation that uses only upwind data. In other words, $v_{j}^{n+1}$ should depend on $v_{j}^{n}$ only for values of $r$ satisfying $r \geq j$. Analyze the order and stability of your method. The method should be at least second-order and stable for all fixed values of $\lambda$ in an interval of the form $(0, C)$, where $C$ is a constant.

[Hint: For the method I came up with, the stability analysis is a mess. I needed to use a symbolic math program (I used the symbolic toolbox in Matlab) to help out. In addition, I needed to help the toolbox with simplification by explicitly substituting double-angle formulas. You can also use a symbolic math program to carry out algebraic simplifications. Hand in printouts in this case. You are also allowed to use the result of Prelim 2, question 2 if that would be helpful.]

Also, it’s OK to use inequalities instead of exact analysis to derive $C$. For example, I can argue that $-2\lambda + 7\lambda^2 + 13\lambda^3 \leq 0$ provided $\lambda \in [0, 0.1]$ since $\lambda \in [0, 0.1]$ implies $7\lambda^2 + 13\lambda^3 \leq 7\lambda(1) + 13\lambda(.01) \leq 2\lambda$.]
2. Show that following conservation-form method for the inviscid Burgers’ equation is $l^1$-contracting:

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}(g(v_j^n, v_{j+1}^n) - g(v_{j-1}^n, v_j^n)),$$

where $g(a, b) = \max(a, 0)^2 + \min(b, 0)^2$. You can impose a CFL condition.

To simplify notation and ease my grading job, use the notation $\hat{a}$ to denote $\max(a, 0)$ and $\hat{a}$ to denote $\min(a, 0)$. Drop the superscript $n$ on the right-hand side in your write-up to ease notation.

[Hint: Note many inequalities like $|\hat{a} - \hat{\hat{a}}| \leq |u - v|$.]

[Notes: (1) This method has appeared in the previous literature. I’ll post the bibliographic citation on the class web-page after the exam is over. (2) One approach to this question, not recommended, is to try to show that this method satisfies a stronger condition than $l^1$-contracting. LeVeque’s book gives an example of a stronger condition. However, since stronger conditions were not covered in lecture, you must also provide a proof that the stronger condition you use implies the $l^1$-contraction condition.]

3. Let $a(T)$ be the aspect ratio of a triangle as defined in lecture. Let $b(T)$ be the product of the two shorter side-lengths of $T$ divided by the area of $T$. (This triangle-quality measure has come up in the finite-element literature.) Show that there is an absolute constant $C$ such that $b(T) \leq Ca(T)$. On the other hand, show that the opposite inequality is not valid. In other words, show that there exists an infinite sequence of triangles $T_1, T_2, \ldots$ such that $a(T_i)/b(T_i) \to \infty$. You are allowed to use the analysis of PS6, Q1 if that would be helpful.

4. Consider the harmonic oscillator from PS2. The equations for this oscillator may be written as the following semi-explicit DAE:

$$p' = -q,$$
$$p^2 + q^2 = r^2,$$

where $r$ is a fixed positive constant. Show that this system has index 1 except possibly at isolated points. Write down the Backward Euler method for this semi-explicit DAE, and solve the nonlinear equations in BE to obtain closed-form formulas for $p^{n+1}, q^{n+1}$. Note that a choice of square-root branch must be made in the formulas. You don’t have to figure out which branch to choose. (Indeed, choosing the branch correctly is a difficulty with this approach, and is related to the existence of non-index-1 points). But you should argue that the method will never take the square root of a negative number, assuming the initial data is real and satisfies the constraint.