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$Ax = b \qquad K_k(A, b) = \{b, Ab, \ldots, A^{k-1}b\}$

	opt problem	subproblem	
MINRES ($A = A^T$)	$\min \|Ax - b\|_2$ s.t. $x \in K_k$	$y_k = \arg\min \|\bar{T}_k y - \beta_0 e_1\|_2$, $x_k = Q_k y_k$	Lanczos: $AQ_k = Q_{k+1}\bar{T}_k$, $\bar{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \beta_{k-1} \\ & \ddots & \ddots & \\ & & \beta_{k-1} & \alpha_k \\ & & & \beta_k \end{pmatrix}$
GMRES	$\min \|Ax - b\|_2$ s.t. $x \in K_k$	$y_k = \arg\min \|\bar{H}_k y - \beta_0 e_1\|_2$, $x_k = Q_k y_k$	$AQ_k = Q_{k+1}\bar{H}_k$, $q_1 = b/\|b\|$, $\bar{H}_k = \begin{pmatrix} \ddots \end{pmatrix}$ ($k+1$ rows, k columns)

$$x = Q_k y$$

$$\|AQ_k y - b\|_2$$

$$= \|Q_{k+1}\bar{H}_k y - b\| \qquad Q_{k+1}^T$$

$$= \left\| \begin{pmatrix} I \\ 0 \end{pmatrix} \bar{H}_k y - \begin{pmatrix} \beta_0 \\ 0 \end{pmatrix} \right\|$$

$$= \|\bar{H}_k y - \beta_0 e_1\|$$

$$\bar{H}_k = G_k \begin{pmatrix} R_k \\ 0 \end{pmatrix}$$

(G_k is $(k+1)\times(k+1)$, R_k is $k \times k$)

$$G_k^T G_k = I$$

$$\|G_k^T \bar{H}_k y - G_k^T \beta_0 e_1\|_2$$

$$= \left\| \begin{pmatrix} R_k \\ 0 \end{pmatrix} y - \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \end{pmatrix} \right\|_2$$

$$R_k y_k = z_k$$

$$\text{residual} = \bar{\eta}_{k+1}$$

$$\bar{H}_{k+1} = \begin{pmatrix} \bar{H}_k & h_{k+1} \\ 0 & h_{k+1,k+1} \end{pmatrix}$$

$$\tilde{G}_k^T \cdots \tilde{G}_1^T \bar{H}_k = \begin{pmatrix} R_k \\ 0 \end{pmatrix}$$

$$\tilde{G}_k^T \cdots \tilde{G}_1^T \beta_0 e_1 = \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \end{pmatrix}$$

$$\tilde{G}_{k+1}^T \tilde{G}_k^T \cdots \tilde{G}_1^T \bar{H}_{k+1} = \begin{pmatrix} R_k & x \\ 0 & x \end{pmatrix}$$

$$\tilde{G}_{k+1}^T \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \\ 0 \end{pmatrix} \to \begin{pmatrix} z_{k+1} \\ \bar{\eta}_{k+2} \end{pmatrix}$$