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Krylov subspace $K_k(A, b) = \{b, Ab, \dots, A^{k-1} b\}$

$\|w\|_A = \sqrt{w^T A w}$

$A \in \mathbb{R}^{n \times n}$ SPD $\Rightarrow$ CG $Ax = b$

$$x_k = \arg\min_x \underbrace{\tfrac{1}{2} x^T A x - b^T x}_{f(x)} \iff x_k = \arg\min_x \| x - \underbrace{A^{-1} b}_{x^*} \|_A$$

s.t. $x \in K_k(A, b)$ s.t. $x_k \in K_k$

First approach: GD on $f(x)$ $x_0, x_1, \dots, x_k, \dots$

$\nabla f(x) = Ax - b$, $g_k = \nabla f(x_k)$ GD: $x_{k+1} = x_k - \alpha_k g_k$

$$f(x_k - \alpha_k g_k) = \tfrac{1}{2}(x_k - \alpha_k g_k)^T A (x_k - \alpha_k g_k) - b^T (x_k - \alpha_k g_k)$$

$$\star \quad = f(x_k) + \tfrac{1}{2} \alpha_k^2 g_k^T A g_k - \alpha_k g_k^T A x_k + \alpha_k b^T g_k$$

$$\frac{\partial}{\partial \alpha_k} \Rightarrow \alpha_k g_k^T A g_k - x_k^T A g_k + b^T g_k$$

$$= 0 \Rightarrow \alpha_k^* = (\underbrace{A x_k - b}_{g_k})^T g_k / g_k^T A g_k = \frac{g_k^T g_k}{g_k^T A g_k}$$

$$f(x_{k+1}) = f(x_k) + \frac{1}{2}\left(\frac{g_k^T g_k}{g_k^T \cancel{A g_k}}\right)^2 g_k^T \cancel{A} g_k - \frac{g_k^T g_k}{g_k^T A g_k} g_k^T g_k$$

$$= f(x) + \frac{g_k^T g_k}{g_k^T A g_k}\left[\frac{1}{2} g_k^T g_k - g_k^T g_k\right] \quad - \frac{1}{2} g_k^T g_k$$

$$g_k^T A^{-1} g_k =$$

$A^{-1}b$

$2b^T x_k$

$+ b^T A^{-1} b$

$$\overset{2}{A} = \frac{1}{2}(x - A^{-1}b)^T A (x - A^{-1}b)$$

big is $T_k(\gamma)$ or small is $1/T_k(\gamma)$?

$$1/T_k(\gamma) \sim \left(\frac{\sqrt{c}+1}{\sqrt{c}-1}\right)^k = \left(1 - \frac{2}{\sqrt{c}-1}\right)^k$$