Nov 13, 2020

Problem: $Ax = b$ today: A symm. pos. def $A = A^T$ $\lambda_1 \geq \dots \geq \lambda_n > 0$

Approach: iterative Krylov subspace method

$$K_k(A, b) = \text{span}\{b, Ab, \dots, A^{k-1} b\}$$

$x_k \in K_k(A, b)$ $x_1, x_2, x_3, \dots$ $\|Ax_k - b\|$

$$f(x) = \frac{1}{2} x^T A x - b^T x \qquad \nabla f(x) = Ax - b \Rightarrow \nabla f(x) = 0 \Leftrightarrow Ax = b$$

Conjugate gradient (CG) method:

$$x_k = \arg\min_x f(x) \quad \text{s.t.} \quad x_k \in K_k(A, b)$$

$\|\cdot\|_M$ $\|x\|_M = x^T M x$ (M SPD)

$$\|r_\cdot\|_{A^{-1}} = \|Ax_\cdot - b\|_{A^{-1}} = (Ax - b)^T A^{-1} (Ax - b) = (Ax - b)^T (x - A^{-1} b) = \underbrace{x^T A x - x^T b - b^T x}_{= 2f(x)} + \cancel{b^T A^{-1} b}$$

$$x_k = \arg\min_x \|Ax - b\|_{A^{-1}} \quad \text{s.t.} \quad x_k \in K_k$$

$\|\cdot\|_2 \rightarrow$ MINRES

Lanczos

Q_k orthonormal basis for K_k

$$AQ_k = Q_{k+1}\bar{T}_k$$

$$\bar{T}_k = \begin{array}{cc} & k \\ k+1 & \begin{pmatrix} \alpha_1 & \beta_1 & \cdots & \beta_{k-1} \\ \beta_1 & \cdots & \beta_{k-1} & \alpha_k \\ & & & \beta_k \end{pmatrix} \end{array}$$

$$Q_k^T A Q_k = T_k = \begin{pmatrix} \alpha_1 & \beta_1 & \cdots & \beta_{k-1} \\ \beta_1 & \cdots & \beta_{k-1} & \alpha_k \end{pmatrix}$$

$$\min \ \tfrac{1}{2} x^T A x - x^T b$$

s.t. $\quad x \in K_k \qquad x = Q_k y \qquad q_1\beta_0$

$$\underbrace{\qquad\qquad}_{T_k} \qquad - y^T \underbrace{Q_k b}_{e_1\beta_0}$$

$$\Rightarrow \quad T_k y = \beta_0 e_1$$

$$x_k = Q_k y_1$$

$O(nk)$ storage for full re-orthog.

$$\|x_{k+1} - x\| \leq \left(1 - \frac{1}{\kappa_2(A)}\right)^{1/2} \|x_k - x\|_2$$