Oct 30, 2020

$A = U\Sigma V^T$

$A^TA = V\Sigma U^TU\Sigma V^T = V\Sigma^2 V^T$

Idea: 1 Unstable!

① form $A^TA = M$

② use eigensolver with M

③ $A^TU = V\Sigma$

$$|\underbrace{\lambda_k(M+E)}_{\tilde{\lambda}_k} - \lambda_k(M)| \le \|E\|_2 \implies |\tilde{\lambda}_k - \lambda_k| = O(\varepsilon_{mach}\|M\|_2) \quad \|M\|_2 = \|A\|_2^2$$

$$|\tilde{\sigma}_k - \sigma_k| \approx \frac{|\tilde{\lambda}_k - \lambda_k|}{\sqrt{\lambda_k}} = O(\varepsilon \|M\|_2/\sigma_k) = O(\varepsilon_{mach}\sigma_1^2/\sigma_k)$$

Want: $|\tilde{\sigma}_k - \sigma_k| = O(\varepsilon_{mach}\cdot\|A\|_2)$

$$S = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{pmatrix}\begin{pmatrix} V \\ U \end{pmatrix}$$

$\|S\|_2 = 2\cdot\|A\|_2$

$$= \begin{pmatrix} V\Sigma \\ U\Sigma \end{pmatrix}$$

$$= \begin{pmatrix} V \\ U \end{pmatrix}\Sigma$$

Idea 2:

① form $S = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$

② eigensolver on S

$$\frac{z^T A z}{z^T z} = \frac{\sum_{i=k}^{n} \lambda_i z_i^2}{\sum_{i=k}^{n} z_i^2} \le \lambda_k$$

$$Qz = Q\begin{pmatrix}0\\ \vdots \\ 0 \\ x \\ \vdots \\ x\end{pmatrix}$$

$$\circledast \le \lambda_k \quad \square$$

Weyl's theorem

A symm with evals $\lambda_1, \ldots, \lambda_n$

$A+E$ symm $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ $\Longrightarrow |\tilde{\lambda}_k - \lambda_k| \leq \|E\|_2$

Cor 1

QR iteration for evals, $A+E$, $\|E\|_2 = O(\varepsilon_{mach}\|A\|_2)$

$$\Longrightarrow |\tilde{\lambda}_k - \lambda_k| \leq O(\varepsilon_{mach}\|A\|_2) = O(\varepsilon_{mach}\max(|\lambda_1|, |\lambda_n|))$$

Cor 2