Oct 9, 2020  LLS: $x = \arg \min_x \|Ax - b\|_2^2$  $A = QR$  $Rx = QTb$

Sensitivity:

1. A nearly singular
2. $b$ not be close to range($A$)

Want: backward stable

$Rx = QTb$

$A = QR$  Is $QR \approx A$?  CGS/MGS

Householder / Givens

Householder  $Q_j = I - 2 v_j v_j^T$

$x = \alpha_j$,  $v_j = \text{sign}(x_j) \|x\|_2 e_1 + x_1$,  $v_j = v_j / \|v_j\|_2$

$Q = Q + E$,  $\|E\|_2 = O(\varepsilon)$

$f((Q+E)A) = f((Q+E)A) = (Q+E)A + F = QA + EA + F$

$= QA + Q^T(EA + Q^T F)$  $(QQ^T = I)$

$\|Q^T EA\|_2 \leq \|Q^T\|_2 \|E\|_2 \|A\|_2 = O(\varepsilon \|A\|_2)$

$\|Q^T F\|_2 \leq \|F\|_2$  $(F_{i,j} = 1_{Q^T \alpha_j \|i\|_2 \|j\|_2} = O(\varepsilon))$  $\|F\|_2 = O(\varepsilon \|A\|_2)$
\( Q(A + G) \quad \|G\|_2 = O(\|E\|_2 \|A\|_2) \quad g = Q^T E A + Q^T F \)

\( f_1(\bar{Q}_2 \bar{A}, A) = f_1(\bar{Q}_2 (Q, A + G)) \quad \|G\|, \|I\|_2 = O(\|E\|_2 \|A\|_2) \)

\[= Q_2(Q, A + G, 1 + G_2) \quad \|G\|_2 = O(\|E\|_2 \|Q_2 (A + G)\|_2) \]

\[= O(\|E\|_2 \|Q_2 (A + G)\|_2) + O(\|E\|) \]

\( (Q_2(Q, A + G, 1 + G_2) + G_2 = Q_2(Q, A + Q_2 G_1 + G_2) \quad O(\|E\|_2 \|A\|_2) \)

\[O(\|E\|_2 \|A\|_2) \]

Multiplying by seq. of orthog matrices is BWS

Cor: \( f_1(\bar{Q}, ... \bar{Q}_n e_j) = q_j + g \quad \|g\| = O(\|E\|_2 \|A\|_2) \)

\( \tilde{Q}^T \tilde{Q} \approx I \)
A is m\times n

\|Ax - b\|_2 \quad \text{A, full rank} \Rightarrow \hat{x} = V\Sigma^{-1}U^Tb

A rank r < n \Rightarrow A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}

\|UEV^T_x - b\|_2 = \|UEV^T_x - U^Tb\|_2

= \| (\Sigma) [V_1^T V_2^T] (x_1) - (u_1^T b) \|_2

= \| (\Sigma) [V_1^T x_1, V_2^T x_2] - (u_1^T b) \|_2

= \| \Sigma, V_1^T x_1 - U^T b \|_2 \quad \hat{x}_1 = V_1 \Sigma^{-1} U^T b

\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \quad \hat{x}_2 = V_2 y \quad A \left( \begin{pmatrix} 0 \\ \hat{x}_2 \end{pmatrix} \right) = 0 \quad V_2 y = \hat{x}_2

V_2 \text{ basis for null } (A) \quad A \left( \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right) = U\Sigma \left( V_1^T \hat{x}_1, V_2^T \hat{x}_2 \right)
\[ \| x \|_2 = \left( \sum_{i=1}^{n} (x_i^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} \]

"Smallest" solution \( x = \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix} \)

no component in null space

\[ A^+ = \sqrt{\Sigma} U^T, \quad \Sigma^+ = \begin{cases} \sqrt{\sigma_i} & \sigma_i > 0 \\ 0 & \sigma_i = 0 \end{cases} \]

What if \( A \) has full rank but \( \sigma_{\min} = k_2(A) \) large?

One idea: drop small singular values "truncated SVD"

\[ A_c = \sqrt{\Sigma_c} U^T \quad \Sigma_c = \begin{cases} \sqrt{\sigma_i} & \sigma_i > \bar{c} \\ 0 & \sigma_i \leq \bar{c} \end{cases} \]

\[ x = A_c^+ b \quad \hat{x} = A_c^+ (b + \delta b) \]

\[ \| x - \hat{x} \|_2 \leq \| A_c^{-1} \|_2 \| \delta b \|_2 \]

\[ \| x - \hat{x} \|_2 \leq \| A_c^{-1} \|_2 \| \delta b \|_2 \leq \frac{\| \delta b \|_2}{\bar{c}} \]

Tradeoff: larger \( \bar{c} \Rightarrow \| A_c x - b \|_2 \) larger
Principal components regression

\[ A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \quad \text{suppose } m \sum_{i=1}^m a_i^T = 0 \]

**PCA:** \[ A = U \Sigma V^T \]

loadings: \[ V_k = [v_1 \ldots v_k] \]

PCs: \[ U_k \Sigma_k = \begin{bmatrix} \delta_1 \sigma_k & \cdots & \delta_k \sigma_k \end{bmatrix} \]

**PCR:** \[ \min \quad \| U_k \Sigma_k x - b \|_2^2 \]

\[ \hat{x} = \Sigma_k^{-1} U_k^T b \]

\[ A_{\Sigma}^T b = V \Sigma_k^+ U_k^T b = U_k \Sigma_k^{-1} U_k^T b = V_k \hat{x} \]

\[ C = \sigma_k + 3 \]
\[ \min_x \| Ax - b \|_2^2 + \lambda^2 \| x \|_2^2 \]

\[ \| (Ax) - (b) \|_2^2 = \| Ax - b \|_2^2 = (\| Ax - b \|_2^2 + \lambda^2 \| x \|_2^2) \]