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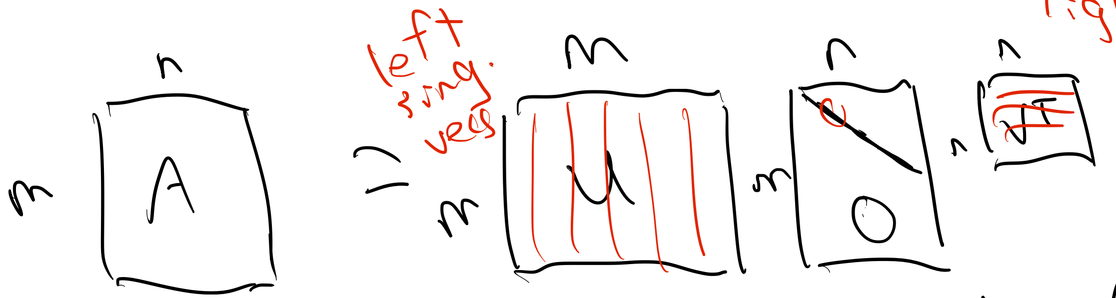
The Singular Value Decomposition (SVD)

$A \in \mathbb{R}^{m \times n}$ $m \geq n$. We can write $A = U \Sigma V^T$

right sing. vecs



$V^T V = I$

$U^T U = I$ $\quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$

singular values

$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n \geq 0$

$$\|A\|_2 = \|\cancel{U} \Sigma \cancel{V^T}\|_2 = \|\Sigma\|_2 = \sup_{\|x\|_2 = 1} \|\Sigma x\|_2^2 \quad \sum_{i=1}^{n} x_i^2 \sigma_i^2 = \sigma_1$$

$$\|A\|_F = \|\Sigma\|_F = \sqrt{\sum \sigma_i^2}$$

SVD existence (sketch)

$\|A\|_2 = \sigma_1 = \sup_{\|x\|_2=1} \|Ax\|_2$ Let $\|Av_1\|_2 = \sigma_1$ $\|v_1\|_2 = 1$

$u_1 = Av_1/\sigma_1$ $\|u_1\|_2 = 1$

$\bar{U} = [u_1 \; U_2]$ $\bar{V} = \begin{array}{cc} 1 & n-1 \\ \end{array}$ $[v_1 \; V_2]$

$$\bar{U}^T A \bar{V} = \begin{array}{c|cc} & 1 & n-1 \\ 1 & \cancel{u_1^T A v_1}\;\sigma_1 & \cancel{0}\;w^T \\ m-1 & \cancel{U_2^T A v_1}\;0 & B \end{array}$$

$u_1^T A v_1 = \sigma_1 u_1^T u_1 = \sigma_1$

$U_2^T A v_1 = \sigma_1 U_2^T u_1 = 0$

$$\left\| \begin{pmatrix} \sigma_1 & w^T \\ 0 & B \end{pmatrix} \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\|_2^2 = \left\| \begin{matrix} \sigma_1^2 + w^T w \\ Bw \end{matrix} \right\|_2^2 \geq \sigma_1^2 + w^T w$$

S

$\|S\|_2 = \|\bar{U}^T A \bar{V}\|_2 = \|A\|_2 = \sigma_1 \Rightarrow w^T w = \|w\|_2^2 = 0$

$\Rightarrow w = 0$

Assume $B = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$ (induction)

$$A = \bar{U} \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix} \bar{V}^T = \underbrace{\bar{U} \begin{bmatrix} 1 & 0 \\ 0 & \underline{U} \end{bmatrix}}_{U} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \underline{\Sigma} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \underline{V}^T \end{bmatrix} \bar{V}^T}_{V^T}$$

unique?