Homework 7, CS 6210, Fall 2020
Instructor: Austin R. Benson
Due December, 16, 2020 at 10:19am ET on CMS (before lecture)

Policies

Collaboration. You are encouraged to discuss and collaborate on the homework, but you have to write up your own solutions and write your own code.
Programming language. You can use any programming language for the coding parts of the assignment. Code snippets that we provide and demos in class will use Julia.
Typesetting. Your write-up should be typeset with LaTeX. Handwritten homeworks are not accepted.
Submission. Submit your write-up as a single PDF on CMS: `https://cmsx.cs.cornell.edu`.

Problems

1. *Augmented Lanczos.*

   Consider the linear system
   $$\begin{bmatrix} \lambda I & A \\ A^T & \lambda' I \end{bmatrix} \begin{bmatrix} s \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$
   where λ and λ' are scalars. Let T_{2k} and Q_{2k} be the matrices generated at step $2k$ of Lanczos for this matrix.

   (a) Show that
   - the diagonal of T_{2k} is $\begin{bmatrix} \lambda & \lambda' & \lambda & \lambda' & \cdots & \lambda & \lambda' \end{bmatrix}$,
   - the super-diagonal of T_{2k} is $\begin{bmatrix} \beta_1 & \beta_1' & \beta_2 & \beta_2' & \cdots & \beta_k \end{bmatrix}$, and
   - $Q_{2k} = \begin{bmatrix} u_1 & 0 & u_2 & 0 & \cdots & u_k & 0 \\ 0 & u_1' & 0 & u_2' & \cdots & 0 & u_k' \end{bmatrix}$,

   where β_j, β_j' and u_j, u_j' are the same for any λ, λ'.

   (b) Show that when $\lambda' = -\lambda$, the vector x solves the regularized least squares problem
   $$\min_x \|Ax - b\|_2^2 + \lambda^2 \|x\|_2^2.$$

   Choosing different values of λ and λ' and deriving CG or MINRES recurrences results in several iterative solvers. For example, LSQR corresponds to using CG for the special case in part (b).

2. *Searching for eigenpairs in Krylov subspaces.*

   Show that $K_k(A, b) = K_k(A - \sigma I, b)$ for any shift σ.

   Thus, if we are looking for eigenvectors in a Krylov subspace, we can find approximations not only of eigenvectors corresponding to the largest-magnitude eigenvalues but to any extremal eigenpairs.

3. *Numerical experiments.*

   Try out some of the built-in iterative solvers for finding the eigenpairs corresponding to the largest or smallest magnitude eigenvalues of a large sparse matrix.

   Here is a Julia code snippet to help you start.

```julia
using Arpack
using SparseArrays

function eigensolver_timings()
    A = sprand(5000, 5000, 0.01)
    k = 10
    # See interface at https://julialinearalgebra.github.io/Arpack.jl/latest/
    t_LM = @elapsed eigs(A, nev=k, maxiter=5000, tol=1e-6, which=:LM)
    t_SM = @elapsed eigs(A, nev=k, maxiter=5000, tol=1e-6, which=:SM)
    return (t_LM, t_SM)
end
```

(a) Does finding the largest or smallest magnitude eigenvalues take longer? Why might this be the case?

(b) Make a plot that shows the running times of the solvers as a function of the number of eigenpairs k.

4. *Course evaluation* **(ungraded)**

Please fill out the course evaluation. Also, feel free to directly email the course staff with feedback.

Thanks for a great semester, and good luck on the final! ☺