Homework 6, CS 6210, Fall 2020
Instructor: Austin R. Benson
Due December, 4, 2020 at 10:19am ET on CMS (before lecture)

## Policies

**Collaboration.** You are encouraged to discuss and collaborate on the homework, but you have to write up your own solutions and write your own code.
**Programming language.** You can use any programming language for the coding parts of the assignment. Code snippets that we provide and demos in class will use Julia.
**Typesetting.** Your write-up should be typeset with LaTeX. Handwritten homeworks are not accepted.
**Submission.** Submit your write-up as a single PDF on CMS: `https://cmsx.cs.cornell.edu`.

## Problems

1. *Lanczos termination.*

   Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $b \in \mathbb{R}^n$. Let $AQ_k = Q_{k+1}\bar{T}_k$ denote the Lanzcos recurrence, as in class. Following the notation in class, let $\beta_k$ be the entry of $\bar{T}_k$ in the last row and last column. Let $\ell$ be the first index $k$ such that $\beta_k = 0$.

   (a) Show that if $A$ is nonsingular, then $A^{-1}b \in K_\ell(A, b)$.
   (b) Show that if $A$ has at most $p$ distinct eigenvalues, then $\ell \leq p$.
   (c) Show that if $A$ is nonsingular, then $A^{-1}b \in K_n(A, b)$.

2. *That norm from class is a norm.*

   Let $M$ be symmetric positive definite. Show that the function $\|\cdot\|_M \colon \mathbb{R}^n \to \mathbb{R}$ given by $\|x\|_M = \sqrt{x^T M x}$ is a norm.

3. *More CG equivalences.*

   Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite and $b \in \mathbb{R}^n$. In class, we showed that the $k$th CG iterate can be interpreted as solving the optimization problems

   $$\text{minimize } \|Ax - b\|_{A^{-1}} \text{ subject to } x \in K_k(A, b)$$

   or

   $$\text{minimize } \frac{1}{2}x^T A x - b^T x \text{ subject to } x \in K_k(A, b).$$

   (a) Show that another equivalent formulation is

   $$\text{minimize } \|A^{-1}b - x\|_A \text{ subject to } x \in K_k(A, b).$$

   (b) Show that another equivalent formulation is

   $$\text{any } x \text{ subject to } x \in K_k \text{ and } Ax - b \perp K_k(A, b).$$

4. *Numerical experiments.*

(a) Implement the three-term recurrence Lanczos algorithm for generating an orthonormal basis for $K_k(A, b)$. Show that the generated vectors can be far from orthogonal.

(b) For this part, we will numerically examine an extension to the result in 1b. Implement a method that generates a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ with $p \ll n$ "clusters" of eigenvalues, where all $n$ eigenvalues are distinct but each is near one of $p$ points. Approximately solve linear systems with a library implementation of CG to show that as the eigenvalues become "more clustered" (more concentrated around the $p$ points), convergence is typically better.