1 Fast Matrix Multiply

Develop the most efficient \( O(n) \) implementation that you can of the following function:

\[
\text{function } y = \text{FastProd}(d,S,T,x) \\
% d and x are column n vectors, S and T are n-by-2, n is a positive power of two. \\
% y = A*x where A = \text{tril}(S*T',-1) + \text{diag}(d) + \text{triu}(T*S',1)
\]

The matrix \( A \) has the property that any submatrix of the form \( A(i_1:i_2,j_1:j_2) \) that does not contain a diagonal entry has rank 2. Efficiency will be measured with \texttt{tic} and \texttt{toc}. You may assume that \( n \leq 2^{20} \). Submit \texttt{FastProd} to CMS.

2 Max Diagonal Entry

Develop the most efficient implementation that you can of the following function:

\[
\text{function } y = \text{MaxDiag}(d,C) \\
% d is column n vectors with positive entries and C is n-by-r with r<<n.
% mu is the largest entry along the diagonal of inv(diag(d) + C*C').
\]

The test script will check things out for \( 100 \leq n \leq 10000 \) and \( 1 \leq r \leq 10 \). Submit \texttt{MaxDiag} to CMS.

3 Rotations

This problem is about rotating one vector into another using a sequence of Givens rotations. (a) Implement the following function so that it performs as specified:

\[
\text{function } [c,s] = \text{TwoVecRotate}(x,\text{alfa}) \\
% x is a nonzero column 2-vector and 0<=\text{alfa}<=\text{norm}(x,2) \\
% c and s satisfy c^2 + s^2 = 1 and have the property that if [c s,-s,c]'*x = y, \\
% then y(1) = \text{alfa}.
\]

(b) Suppose \( c^2 + s^2 = 1 \) and \( 1 \leq i < j \leq n \). Let \( G(c,s,i,j) \in \mathbb{R}^{n\times n} \) denote the \( n \)-by-\( n \) Givens rotation in planes \( i \) and \( j \), e.g.,

\[
G(c,s,2,4) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & c & 0 & s & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -s & 0 & c & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

By making effective use of \texttt{TwoVecRotate(x,alfa)} implement the following function so that it performs as specified:

\[
\text{function } [c,s,idx,jdx] = \text{Uniform}(v) \\
% v is a column n-vector with unit 2-norm. \\
% c and s are column (n-1)-vectors with c.^2 + s.^2 = \text{ones}(n-1,1) and \\
% idx and jdx are column (n-1)-vectors of integers with the property that \\
% 1<= idx(k)<jdx(k)<=n, k=1:n-1 and \\
% G(c(n-1),s(n-1),idx(n-1),jdx(n-1))'*...*G(c(1),s(1),1,idx(1),jdx(1))' * v = \text{ones}(n,1)/\text{sqrt}(n)
\]

Submit \texttt{TwoVecRotate} and \texttt{Uniform} to CMS.