CS 621: Take-Home Midterm

Due: Friday October 25, 2002 (In Upson 4130 By 4pm)

The only discussion allowed is with cv@cs.cornell.edu. Submit a hardcopy of each (fully commented) solution function in Upson 4130 by 4pm on Friday the October 25th. Include the proof required in 3(a) and any other "derivation notes" that may be useful, especially if you couldn't fully solve a problem. Electronic submissions of the four required MATLAB functions are also needed. Send a single email to Siddharth at alexande@cam.cornell.edu with the four m-files included as attachments.

1. How would you solve a nonsingular linear system of the form

$$\begin{bmatrix} A & B \\ uv^T & A^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} ?$$

Here, $A \in \mathbb{R}^{n\times n}$ is nonsingular, $B \in \mathbb{R}^{n\times n}$, and $u, v, c, d, y, z \in \mathbb{R}^n$. Express your answer in the form of a MATLAB function `[y,z] = BlockSolve(A,u,v,c,d)`. (All vector parameters should be column vectors.) Your implementation must make effective use of the MATLAB `lu`.

2. Suppose $A \in \mathbb{R}^{m\times n}$ has rank m and that $m < n$. There are an infinite number of solutions to the underdetermined linear system $Ax = b$ where $b \in \mathbb{R}^m$. How could you compute the solution that minimizes $\| x - c \|_2$ where $c \in \mathbb{R}^n$ is given? Express your answer in the form of a MATLAB function

```
function x = SolveAndClose(A,b,c)
```

The vectors `b`, `c`, and `x` should be column vectors. Your implementation must make effective use of the MATLAB `qr` function.

3. Suppose

$$A = \begin{bmatrix} 0 & -\alpha & -u^T \\ \alpha & 0 & -v^T \\ u & v & B \end{bmatrix}$$

is an n-by-n skew-symmetric matrix and that $u, v \in \mathbb{R}^{(n-2)}$ and $B \in \mathbb{R}^{(n-2)\times(n-2)}$. If $\alpha \neq 0$ then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -v/\alpha & u/\alpha & I_{n-2} \end{bmatrix} \begin{bmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix} \begin{bmatrix} 1 & 0 & -v^T/\alpha \\ 0 & 1 & u^T/\alpha \\ 0 & 0 & I_{n-2} \end{bmatrix} \qquad (1)$$

where

$$\tilde{B} = B + \frac{1}{\alpha}\left(uv^T - vu^T\right).$$

Note that this matrix is also skew-symmetric so it too can be factored as in (1) provided $\tilde{b}_{21} \neq 0$. If $n = 2m$ then repetition of this process would culminate in the production of a factorization $A = LDL^T$ where $L \in \mathbb{R}^{2m\times 2m}$ is unit lower triangular and D is a direct sum of 2-by-2 skew-symmetric matrices,

$$D = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_m \end{bmatrix}.$$

This assumes that no zero α's are encountered along the way. For things to be numerically stable (without pivoting) we must be able to guarantee that quotients like v/α and u/α are "nice". To that end we define a generalized type of diagonal dominance for skew matrices. We say that the skew-symmetric matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \qquad A_{ij} \in \mathbb{R}^{2\times 2}$$

is block diagonally dominant if

$$\sum_{i \neq j} \| A_{ij} \|_1 < \| A_{jj} \|_1 \qquad j = 1{:}m$$

(a) Prove that if this is the case then the entries in L are bounded by 1 in absolute value. Hint: show that if A is block diagonally dominant in (1) then so is $\tilde{B}$. (b) Implement the following MATLAB function.

```
  function [L,D] = SkewFactor(A)
% A is a 2m-by-2m block diagonally dominant skew symmetric matrix.
% A = LDL' where L is a 2m-by-2m unit lower triangular matrix and
% D is an m-by-m block diagonal matrix with 2-by-2 blocks.
```

Your implementation should be vectorized and flop-efficient.

4. Suppose $R_1, \ldots, R_{m-1} \in \mathbb{R}^{p \times p}$ are given symmetric matrices and that

$$T = \begin{bmatrix} I_p & R_1 & R_2 & \cdots & \cdots & R_{m-2} & R_{m-1} \\ R_1 & I_p & R_1 & R_2 & \cdots & \vdots & R_{m-2} \\ R_2 & R_1 & I_p & R_1 & \ddots & \vdots & \vdots \\ \vdots & R_2 & R_1 & I_p & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & R_2 \\ R_{m-2} & \cdots & \cdots & \ddots & \ddots & I_p & R_1 \\ R_{m-1} & R_{m-2} & \cdots & \cdots & R_2 & R_1 & I_p \end{bmatrix}$$

is positive definite. Recognize this as a block Toeplitz matrix. Develop a fast method for solving

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where $b_1, \ldots, b_m \in \mathbb{R}^p$ are given and $x_1, \ldots, x_m \in \mathbb{R}^p$ are sought. Try to generalize the Durbin and Levinson algorithms. Make use of the "block exchange" matrices, i.e., matrices that look like

$$E = \begin{bmatrix} 0 & 0 & 0 & I_p \\ 0 & 0 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \end{bmatrix}$$

The generalized Yule-Walker systems might look something like this

$$\begin{bmatrix} I_p & R_1 & R_2 & R_3 \\ R_1 & I_p & R_1 & R_2 \\ R_2 & R_1 & I_p & R_1 \\ R_3 & R_2 & R_1 & I_p \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = - \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

where the Y_i are also p-by-p. Implement the following MATLAB function

```
  function x = BlockLevinson(R,b)
% R is a length m-1 cell array with R{i} a p-by-p symmetric matrix.
% The block Toeplitz matrix T defined by R_{1},...,R{m-1} is positive definite.
% b is a length m cell array with each b{i} a column p-vector.
% x is a length m cell array with each x{i} a column p-vector with
% the property that T[x{1};...;x{m}] = [b_{1};...;b{m}].
```