are since constructive completeness with respect to full intuitionistic validity contradicts *Church's Thesis* [49,80] and implies *Markov's Principle* as well [60,61].

1.7. Constructive type theory with an intersection operator

We first informally discuss *evidence semantics* for minimal logic.[7] Using evidence semantics, we introduce the idea of *uniform validity*, a concept central to our results and one that is also classically meaningful.

This concept provides an effective tool for constructive semantics because we can establish uniform validity by *exhibiting even one polymorphic object* among a possibly unbounded number of them that we might find. For example, the propositional formula $A \Rightarrow A$ is uniformly valid exactly because there is an object in the intersection of the family of all evidence types for this formula indexed by each possible choice of proposition A among the *type of propositions*, $\mathbb{P}$.

We write this intersection as $\forall[A : \mathbb{P}].A \Rightarrow A$ or as $\bigcap A : \mathbb{P}.A \Rightarrow A$.[8] In this case, given the extensional equality of functions, the polymorphic identity function $\lambda(x.x)$ is the one and only object in the intersection. So the witness for uniform validity, like the witness for provability, can be provided by a single object. Truth tables provide a single (expensive) piece of evidence for classical propositional logic. There are single witnesses for the validity of all uniformly valid first-order formulas. For example, it will be clear after we provide the evidence semantics that the polymorphic term $\lambda(h.\lambda(x.\lambda(p.h(\langle x,p\rangle))))$ establishes the uniform minimal (logic) validity of

$$\sim \exists x.P(x) \Rightarrow \forall x.(\sim P(x))$$

hence the uniform intuitionistic and classical validity as well.

Another important observation about uniform validity is that *the formulas of first-order logic that are provable intuitionistically and minimally are uniformly valid.* It is also noteworthy that *the law of excluded middle is not uniformly valid in either constructive or classical evidence semantics.*

The meaning of *False* also raises the semantic issue that leads us to first consider minimal logic and the Friedman embedding of *iFOL* into *mFOL*. Consider the intuitionistically valid assertion *False* $\Rightarrow A$ for any proposition A. One type theory witness for uniform validity is $\lambda(x.x)$, and other witnesses include any constant function, say $\lambda(x.17)$. If we designate a *diverging* term such as *div*, then $\lambda(x.div)$ is also evidence because the claim being made is that if x belongs to the evidence type for *False*, then x or 17 or *div* belongs to the evidence type for A. So according to the informal semantics of intuitionistic logic, the claim *False* $\Rightarrow A$ is "vacuously true" since no element can be evidence for *False* whose standard evidence is the empty type.

From the constant function with an arbitrary evidence term *evd*, $\lambda(x.evd)$, we cannot reconstruct the proof of *False* $\Rightarrow A$. The term *evd* might be entirely misleading. The evidence in the constructive metatheory, *CTT*, for *False* $\Rightarrow A$ provided by the *CTT* proof is $\lambda(x.any(x))$. This evidence suggests a way to provide an alternative semantics for *False*, and thus for *iFOL*, that avoids the issue just discussed and avoids the need for minimal logic in our account. On the other hand, the minimal logic approach is extremely simple, and it is well known and well studied. So we use that method first and then point out how to avoid it.

We are using the new semantics of *False* and *iFOL* in our Nuprl proof, and we will account for it much more fully in a future article about the formalization of our proof in *CTT*.

In minimal logic, there is no atomic propositional constant *False*. Instead the *arbitrary* propositional constant $\perp$ is used, and its interpretation allows non-empty types as well as empty ones. For the same

[7] We can extend this semantics to classical logic if oracle computations are allowed to justify the law of excluded middle, $P \vee \sim P$, with an operator $magic(P)$ [19]. We make some observations about classical logic based on this *classical evidence semantics*.

[8] We work in a *predicative* metatheory, therefore the type of all propositions is stratified into orders or levels, written $\mathbb{P}_i$. For these results we can ignore the level of the type or just write $\mathbb{P}$.

reason, avoiding vacuous hypotheses, we require that all domains of discourse for minimal logic can be non-empty.

1.8. Comparison to intuitionistic validity

Results of McCarty [60,61] demonstrate that unless one changes in significant ways what one means by completeness (or by validity) or otherwise limits the collection of formulas at issue, then a completeness theorem will be impossible to prove intuitionistically. We have opted to change the notion of validity, not by preconceived choice, but by a discovery.

We discovered that provability is captured exactly by *uniform validity*, an intuitively smaller collection of formulas than those constructively valid. Nevertheless, uniform validity is extremely useful in practice when thinking about purely logical formulas precisely because it corresponds exactly to proof and yet is an entirely semantic notion based on evidence semantics, the semantics that enables strong connections to computer science.

1.9. Counterexamples

Soundness with respect to uniform validity provides a simple method of showing that formulas are not provable by showing that they are not uniformly valid. For example, it is trivial to show that $P \vee {\sim}P$ is not uniformly valid, e.g. to show $\sim \forall[P : \mathit{Prop}].P \vee {\sim}P$. Suppose there were a uniform realizer, d. It would have to be an element of the disjoint union type, thus either $inr(\star)$ or $inl(\star)$. If it is an inr term, then pick P to be a true proposition, say $True$, and otherwise pick it to be *False*. These choices show that there can be no such uniform d. Once we have this easy result, we can show that ${\sim\sim}\, P \Rightarrow P$ is also not uniformly valid, e.g. we show $\sim \forall[P : \mathit{Prop}].\ {\sim\sim}\, P \Rightarrow P$. We do this by assuming that $\forall[P : \mathit{Prop}].\ \sim {\sim}P \Rightarrow P$ and using the fact that for any P we can prove ${\sim\sim}\,(P \vee {\sim}P)$, thus if we could prove the uniform statement, we could also prove $\forall[P : \mathit{Prop}].P \vee {\sim}P$ which we just showed is not uniformly true. By the same technique we can show that Pierce's law is not uniformly valid, e.g. $\sim \forall[P, Q : \mathit{Prop}].(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P)$. All the formulas that intuitionistically imply $P \vee {\sim}P$ are provably false by this method.

We can also show that first-order Markov's Principle (MP) is not uniformly valid; that is $(\forall x.(P(x) \vee {\sim}P(x))\& \sim \forall x.{\sim}P(x)) \Rightarrow \exists y.P(y)$ is not uniformly valid. This is because we can choose a two element domain D with elements a and b and consider two predicates, P_1 and P_2 which have opposite values on a and b. The existential quantifier must pick one of a or b, but it will be an incorrect choice for one of the predicates.

2. Proof rules and proof expressions

2.1. Proof expressions

We assign denotational meaning to proofs, according to the "proofs-as-terms" principle (PAT). The rules include constraints on the subexpressions of a proof. This is especially natural in *refinement style logics* studied by Bates [6] and Griffin [31] and used in *CTT* and *tableaux systems* [12,72,27].

For each rule we provide a name that is the *outermost operator* of a proof expression with slots to be filled in as the refinement style proof is developed. The partial proofs are organized as a tree generated in two passes. The first pass is top down, driven by the user creating terms with slots to be filled in on the algorithmic bottom up pass once the downward pass is complete. Here is a simple proof of the intuitionistic tautology $A \Rightarrow (B \Rightarrow A)$.

$$\vdash A \Rightarrow (B \Rightarrow A)\ by\ \lambda(x.slot_1(x))$$