Inconsistency of Primitive Recursive Arithmetic

Edward Nelson

With an

Introduction by Sarah Jones Nelson

and an

Afterword by Sam Buss and Terence Tao
Introduction

In September of 2011 Edward Nelson announced that he had a proof of the inconsistency of Peano Arithmetic. He had devoted twenty-five years to constructing the proof. When Terence Tao and Daniel Tausk independently found an error, Ed withdrew his claim at once and cheerfully returned to work on it the next day. By March of 2013 – confident that he had corrected the error – he wrote a project proposal with “a crucial new insight: technically, bounds in the Hilbert-Ackermann consistency theorem depending only on rank and level, but not on the length of the proof.” The project he proposed “challenges the entire current understanding and practice of mathematics....It will radically change the way mathematics is done. This will affect the philosophy of mathematics, how the nature of mathematics is conceived. It will also affect the sciences that use mathematics, especially physics....It addresses a very big question: does mathematics consist in the discovery of truths about some uncreated eternal reality (the traditional Platonic view), or is it a humble human endeavor to construct abstract patterns that will be sound, free of all contradiction, in the hope that some will be beautiful, uplifting the human spirit, and that some (not necessarily different ones) will be of practical use to improve the human lot?”

The following excerpts from Ed’s proposal describe his vision of a new mathematics and the open question of the consistency of Peano Arithmetic.

“Peano Arithmetic is one of the simplest and most fundamental of mathematical theories. Its consistency, however, has not been proved by any means that all mathematicians accept. It implies that all primitive recursive functions are total, but they are directly defined only for numerals, and the argument that the values always reduce to numerals is circular. The proposal is to complete a proof that Peano Arithmetic is in fact inconsistent. The principal output will be a book entitled ‘Elements’. The outcome will be a major change in the way mathematics is done, with philosophical and scientific consequences.

“The guiding spirit of this investigation is that mathematics is not some uncreated abstract reality that we can take for granted and explore, but that it is a human endeavor in which one should begin by looking at the very simplest concepts without taking them for granted.

“Numbers are constructed from 0 by successively taking successors; S...S0 is called a numeral. Definitions of primitive recursive functions, such as addition, multiplication, exponentiation, superexponentiation, and so forth, are schemata for constructing numerals. They define a value for 0 and then a value for Sx in terms of the value for x. But when numerals are substituted for the variables in such a schema it is not clear that it defines a numeral: the putative number of steps
needed to apply the definitions can only be expressed in terms of the expressions
themselves. The argument is a vicious circle. Consequently, the consistency of
Primitive Recursive Arithmetic, and \textit{a fortiori} of Peano Arithmetic (P), is an open
question.

“Here is a nontechnical description of how I propose to show that P is inco-
sistent. We start with a weaker theory Q and by relativization techniques extend
it to a stronger theory Q^* . Proofs in Q^* reduce to proofs in Q. Q^* arithmetizes Q
itself – that is, it expresses the syntax of Q by a term of Q^* that here I shall denote
by @Q. Remarkably, Q^* proves that there is no open proof of a contradiction in
@Q. (‘Open’ means that the proof has no quantifiers, i.e., symbols for ‘there exists’
and ‘for all’.) All of this was done in my book ‘Predicative Arithmetic’ (Prince-
ton University Press, 1986) and is being redone with complete proofs in the book
‘Elements’, a work in progress that is the subject of this proposal. The Hilbert-
Ackermann consistency theorem implies that there is no proof of a contradiction,
even with quantifiers, in @Q. This theorem can be only partially established in
Q^*. The crucial new insight is that it can be proved provided there are bounds on
features of the proof (called rank and level) but emphatically not depending on the
length of the proof. These bounds on rank and level cannot be proved in Q^*, but
for each specific proof, P proves that Q^* proves them! This opens the way to exploit
the stunning proof without self-reference of Gödel’s second incompleteness theorem
by Kritchman and Raz (Notices of the American Mathematical Society, December
2011). The upshot is that P proves that Q is inconsistent. But, as is well known,
P also proves that Q is consistent. Therefore Peano Arithmetic P is inconsistent.”

Ed was the only living mathematician who could argue from purely syntactic
reasoning without the traditional semantics established by Plato. John Conway
suggested to me that this might explain why no one has fully understood Ed’s
depthly unique insights into the foundations of contemporary mathematics. He stood
alone in the world, courageous as a formalist of a new ontology of integers: proof
that completed infinities do not exist and that human minds invented numbers
never discovered or revealed from platonic forms of any fundamental reality. I am
hopeful that the mathematical community will boldly investigate “Elements” and
the unshakeable foundations Ed sought to build.

Sarah Jones Nelson
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Photo courtesy Sarah Jones Nelson

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1. Against finitism

Primitive Recursive Arithmetic (PRA) was invented by Skolem [Sk] in response to
Principia Mathematica with the express purpose of avoiding quantification over infinite
domains. His tools were primitive recursion and induction.

Familiar examples of primitive recursions are

\[
\begin{align*}
    x + 0 &= x \\
    x + Sy &= S(x + y) \\
    x \cdot 0 &= 0 \\
    x \cdot Sy &= x + (x \cdot y) \\
    x \uparrow 0 &= S0 \\
    x \uparrow Sy &= x \cdot (x \uparrow y) \\
    x \uparrow 0 &= S0 \\
    x \uparrow Sy &= x \uparrow (x \uparrow y)
\end{align*}
\]

A numeral is a term containing only S and 0, and a primitive recursive (PR) number
is a variable-free term all of whose function symbols are PR. The finitary credo is that
PR numbers reduce to numerals by applying the equations a sufficient number of times.
If indeed that were so, the applications used could be counted by a numeral. But in
general the number of applications needed can be expressed only in terms of PR numbers
themselves—the argument is blatantly circular.

The objection being raised here is not some vague semantic “ultrafinitistic” assertion
that some PR numbers are so big they don’t really exist. Certainly the PR number
SS0 \uparrow SSSSS0 exists: here it is, in front of our eyes, consisting of eleven symbols. The
problem is syntactical. Let A be a formula that holds for 0 and is such that whenever it
holds for x it holds for Sx. Then A holds for any numeral n; this follows from the basis
A_x(0) by modus ponens applied as many times as there are occurrences of S in n. But the
postulation of induction, implying that A holds for every PR number, is an expression of
the finitary credo.

PRA is a boldly speculative attempt to treat PR numbers as if they were equal to
numerals. We shall see that it is inconsistent.

2. Outline

The next section describes the notational and terminological conventions used in this
paper (including the present outline) and formulates PRA as a formal system. Section 4
explicitly defines a binary function symbol Eq such that ⊢ Eq(x, y) = 0 ⇔ x = y and
§5 introduces bounded quantifiers. Section 6 introduces strings and their combinatorics,
and §7 uses them to formulate arithmetization. Section 8 establishes a form of Chaitin’s
The heart of the paper is §9, which constructs a subsystem of PRA that proves the consistency of its own arithmetization. The final §10 exploits the method of the Kritchman-Raz proof www.ams.org/notices/201011/rtx101101454p.pdf, based on Chaitin’s theorem and the surprise examination paradox, together with the self-consistency result of §9, to derive a contradiction in PRA.

Numbered and italicized steps occur throughout the paper to serve as a fuller outline.

3. The formal system PRA

#1. Formulate the axioms and rules of inference of PRA.

The symbols of PRA are variables, function symbols, =, ¬, and ∨. A “decorated letter” is a letter with zero or more digits as subscript and zero or more primes as superscript. We use decorated s to stand for symbols. Decorated italic letters are variables, and decorated x y z w stand for variables. An expression is a concatenation of symbols; decorated u v stand for expressions, f g h for function symbols. Each symbol s has an index (or arity), denoted by ιs, specifying how many arguments it takes. A symbol is 0-ary, unary, binary, and so forth, according as its index is 0, 1, 2, and so forth. A constant is a 0-ary function symbol; decorated e stands for constants. Variables are 0-ary, ¬ is unary, = and ∨ are binary. Terms are defined recursively as follows: x is a term; if u1, . . . , uιf are terms then fu1 . . . uιf is a term. Decorated a b c d stand for terms. An equation is =ab. Formulas are defined recursively as follows: equations are formulas; if u and v are formulas, so are ¬u and uv. (There are no quantifiers in PRA.) Decorated A B C D H stand for formulas. We frequently use infix notation for binary symbols; thus a = b abbreviates =ab and A ∨ B abbreviates ∨AB. The use of infix notation often requires groupers to avoid ambiguity; we use parentheses to group terms, and brackets and braces to group formulas. Some other useful abbreviations are A & B for ¬[¬A ∨ ¬B], A → B for ¬A ∨ B, and A ↔ B for [¬A ∨ B] & [A ∨ ¬B]. The symbol ¬ binds tightly, and ∨ and & bind more tightly than → and ↔. Apart from these precedence rules, infix symbols are associated from right to left. Function symbols are nonlogical symbols.

If ℓ is a decorated letter, ℓ⃗ abbreviates ℓ1 . . . ℓµ for some µ called the multiplicity of ℓ⃗ and denoted by µ⃗ℓ. The notation a ̸= b abbreviates ¬a = b. Let u⃗x(⃗a), where µ⃗x = µ⃗a, be the expression, called an instance of u, obtained by replacing each occurrence of xν in u by aν, for all ν with 1 ≤ ν ≤ µ⃗x. Whenever we write u⃗x(⃗a) it is understood that µ⃗x = µ⃗a.

Sometimes parentheses and commas are inserted into terms to enhance readability. Although A⃗x(x) and A are the same, the redundant notation A⃗x(x) often increases readability.

A truth valuation on A is a function τ from the equations in A to {T, F}. We extend τ to all subformulas of A, keeping the notation τ, by letting τ(¬B) be T if and only if τ(B) is F, and letting τ(B ∨ C) be T if and only if τ(B) is T or τ(C) is T. A tautology is a formula A such that τ(A) is T for all truth valuations τ on A; A is a tautological consequence of A1, . . . , Aν in case A1 → · · · → Aν → A is a tautology. Call A and B tautologically equivalent in case A ↔ B is a tautology. (Since → is associated from right to left, A1 → · · · → Aν → A is tautologically equivalent to A1 & · · · & Aν → A.)

A numeral is a term containing no symbols other than S and 0.
Let $x, y,$ and $z$ be distinct, let $a$ contain no variables other than those in $x$, and let $b$ contain no variables other than those in $x, y,$ and $z$. Then $f$ is defined by primitive recursion from $a$ and $b$ by

A1. $f(x, 0) = a$
A2. $f(x, S(y)) = b_z(f(x, y))$

A construction of $f$ is a finite sequence $g_1, g_2, \ldots, g_\nu$ where $g_1$ is 0, $g_2$ is $S$, $f$ is in the sequence, and each $g_\mu$ for $3 \leq \mu \leq \nu$ is constructed by primitive recursion from terms containing no function symbols other than those strictly preceding it in the sequence. The PR function symbols are those that have a construction. A PR term is a term in which every function symbol is PR, and a PR number is a variable-free PR term.

PRA is formulated as a formal system as follows. Its nonlogical symbols are the PR function symbols. Its nonlogical axioms are the construction axioms A1 and A2 for definitions of PR function symbols, and the successor axioms

a3. $\neg Sx = 0$
a4. $Sx = Sy \rightarrow x = y$

The logical axioms are reflexivity

a5. $x = x$
symmetry
a6. $x = y \rightarrow y = x$
the equality axioms
A7. $x = y \land A_x(x) \rightarrow A_x(y)$

and the propositional axioms
A8. $A \lor A \rightarrow A$
A9. $A \rightarrow A \lor B$
A10. $A \lor B \rightarrow B \lor A$
A11. $[B \rightarrow C] \rightarrow [A \lor B \rightarrow A \lor C]$

There are three rules of inference:

instance: from $A$ infer an instance of $A$
modus ponens: from $A$ and $A \rightarrow B$ infer $B$
induction: from $A_x(0)$ and $A_x(x') \rightarrow A_x(Sx')$ infer $A_x(x)$

As in any formal system, a proof is a finite sequence of formulas each of which is either an axiom or follows from strictly preceding formulas by a rule of inference, and it is a proof of $A$ in case $A$ is in the sequence. Decorated $\pi$ stands for a finite sequence of formulas. The notation $\pi \vdash A$ means that $\pi$ is a proof in PRA of $A$, while $\vdash A$ means that there is a proof $\pi$ in PRA of $A$, in which case $A$ is a theorem of PRA.

The Propositions in this paper are metamathematical in nature; they are statements about PRA whose proofs are finitary in the strict sense of being expressible in PRA.
PROPOSITION 1. Certain familiar devices can be used to extend the notion of proof.

(i) Tautologies are theorems of PRA.

(ii) A tautological consequence of theorems of PRA is a theorem of PRA.

(iii) Previously proved theorems may be cited in proofs.

(iv) Deductions may be used in proofs, as follows. Introduce, in the course of a proof, an arbitrary formula $H$, the hypothesis, and follow it by $B_1, \ldots, B_\nu$ where each $B_\mu$ for $1 \leq \mu \leq \nu$ is a theorem or is a tautological consequence of strictly preceding formulas. Then discharge the hypothesis by writing $H \rightarrow B_\nu$ and never using $H, B_1, \ldots, B_\nu$ again.

(v) Claims may be established, as follows. State a claim $A$, introduce the hypothesis $\neg A$, and follow it by $B_1, \ldots, B_\nu$ where these are as in (iii) and furthermore a contradiction is obtained, meaning that for some preceding $B$ is $\neg B$. Then establish the claim by writing $A$ and never using $\neg A, B_1, \ldots, B_\nu$ again.

Proof. For (i), refer to [HA]. Hilbert and Ackermann give a finitary proof that in the formal system whose only axioms are 8–11 and whose only rule of inference is modus ponens, the theorems are precisely the tautologies.

For (ii), suppose that $\vdash A_1, \ldots, A_\nu$ and that $A_1 \rightarrow \cdots \rightarrow A_\nu \rightarrow A$ is a tautology, and hence a theorem by (i). Then we have $\vdash A_2 \rightarrow \cdots \rightarrow A_\nu \rightarrow A$ by modus ponens. Proceeding in this way we obtain a proof of $A$ in $\nu$ steps. In other words, tautological consequence is a derived rule of inference in PRA.

For (iii), just insert the proofs of the cited theorems.

Given a deduction as in (iv), replace $H, B_1, \ldots, B_\nu$ by $H \rightarrow H, H \rightarrow B_1, \ldots, H \rightarrow B_\nu$. Then $H \rightarrow H$ is a theorem by (i). If $B_\mu$ is a theorem then $H \rightarrow B_\mu$ is a tautological consequence of it. If $B_\mu$ is a tautological consequence of strictly preceding formulas, then $H \rightarrow B_\mu$ is a tautological consequence of them with the $B_\lambda$, for $1 \leq \lambda < \mu$, among them replaced by $H \rightarrow B_\lambda$. In this way, by (ii), we have a proof of $H \rightarrow B_\nu$, proving (iv). (Notice that no instance of $H$ is taken. This is sometimes expressed by saying that the variables in $H$ are held constant.)

Given a claim as in (v), proceed as in (iv). Any formula, in particular $A$, is a tautological consequence of $B_\nu$ and $\neg B_\nu$, so adjoin $A$ to the deduction. Discharging the hypothesis $\neg A$ we obtain $\neg A \rightarrow A$, of which $A$ is a tautological consequence, proving (v). (A special case of this is an indirect proof, in which the theorem itself is the claim.)

The only predicate symbol in PRA is $=$. Nevertheless, we can introduce other predicate symbols as abbreviations. Given $A$, let $\bar{x}$ be its distinct variables in some order, set $p(\bar{x}) \leftrightarrow A$, and let $p(\bar{a})$ abbreviate the instance $A_\bar{x}(\bar{a})$. Use decorated $p\, q$ to stand for predicate symbols other than $=$; they occur only in abbreviations. An explicit definition of $f$ is $f(\bar{x}) = c$ where no variable other than those in $\bar{x}$ occurs in $c$. Then let $f(\bar{a})$ abbreviate $c_\bar{x}(\bar{a})$. Explicitly defined function symbols occur only in abbreviations.

Some formulas are marked ★ or ★★ for emphasis.

4. Equality

#2. Construct Eq so that $\vdash Eq(x, y) = 0 \leftrightarrow x = y$.

Hilbert and Bernays construct such a function symbol in §7 of the first edition of [HB] (1939). Following a suggestion of Kreisel to establish the basic properties of $<$ without
using addition, Bernays omitted this explicit construction in the second edition (1968). We give the surprisingly long construction of Eq from the first edition. (Their unary \( \delta \) is our \( P \), their binary \( \delta \) is our \( - \), and their \( \delta(a, b) + \delta(b, a) \) is our Eq\((a, b)\).)

Primitive recursions are labeled with \( r \), theorems with \( t \), explicit definitions with \( e \), and definitions of PR predicate symbols with \( d \); if one of these letters is capitalized, it indicates a schema. If \( \xi \) labels a theorem \( A \), and \( x_1, \ldots, x_\nu \) are the first \( \nu \) distinct variables of \( A \) in the order of first occurrence, then \( \xi; a_1; \ldots; a_\nu \) is the theorem \( A_{x_1,\ldots,x_\nu}(a_1, \ldots, a_\nu) \).

\[ r_{12}. \quad x + 0 = x \quad \& \quad x + Sy = S(x + y) \]

\[ t_{13b}. \quad 0 + 0 = 0 + 0 \]

**Proof.** H: 5;0+0

\[ t_{13i}. \quad x + 0 = 0 + x \rightarrow Sx + 0 = 0 + Sx \]

**Proof.** H:x 12;Sx 12;0;x 12;x

This is an indirect proof. The theorem being proved is a disjunction, \( \neg x + 0 = 0 + x \vee Sx + 0 = 0 + Sx \), and H:x is its negation \( x + 0 = 0 + x \& \neg Sx + 0 = 0 + Sx \) (with \( \neg \neg \) removed). The colon indicates that \( x \) is to be held fixed with this introduction of a hypothesis. The remaining formulas in the text proof together with some equality substitutions (instances of equality axioms A5) and implicit uses of symmetry a6, give a contradiction (marked QEA for quod est absurdum) by tautological consequence, completing the indirect proof of the theorem. If you are reading this online (it is posted at not yet www.math.princeton.edu/~nelson/papers/Balrog.pdf), click on the blue **Proof** link. Otherwise, open a browser to www.math.princeton.edu/~nelson/proof/ and click on 13i.pdf.

\[ t_{13}. \quad x + 0 = 0 + x \]

When \( t\xi \) immediately follows \( t\xi b \) and \( t\xi i \), it is an inference by induction.

\[ t_{14b}. \quad x + S0 = Sx + 0 \]

**Proof.** H:x 12;x:0 12;x 12;Sx

\[ t_{14i}. \quad x + Sy = Sx + y \rightarrow x + SSy = Sx + Sy \]

**Proof.** H:x:y 12;x:Sy 12;Sx;y

\[ t_{14}. \quad x + Sy = Sx + y \]

\[ t_{15b}. \quad x + 0 = 0 \rightarrow x = 0 \& 0 = 0 \]

**Proof.** H:x 12;x

\[ t_{15i}. \quad [x + y = 0 \rightarrow x = 0 \& y = 0] \rightarrow [x + Sy = 0 \rightarrow x = 0 \& Sy = 0] \]

**Proof.** H:x:y 12;x:y 3;x+y

\[ t_{15}. \quad x + y = 0 \rightarrow x = 0 \& y = 0 \]

\[ r_{16}. \quad P0 = 0 \& PSx = x \]

\[ r_{17}. \quad x - 0 = x \quad \& \quad x - Sy = P(x - y) \]
t18b. \( Sx - S0 = x - 0 \)
Proof. \( H: x \quad 17;Sx;0 \quad 17;Sx \quad 16;x \quad 17;x \)

t18i. \( Sx - Sy = x - y \rightarrow Sx - SSy = x - Sy \)
Proof. \( H: x; y \quad 17;Sx;Sy \quad 17;x;y \)

4. \( Sx - Sy = x - y \)

t19b. \( 0 - 0 = 0 \)
Proof. \( H \quad 17;0 \)

t19i. \( x - x = 0 \rightarrow Sx - Sx = 0 \)
Proof. \( H: x \quad 18;x;x \)

t19. \( x - x = 0 \)

t20b. \( S0 - 0 = S0 \)
Proof. \( H \quad 17;S0 \)

t20i. \( Sx - x = S0 \rightarrow SSx - Sx = S0 \)
Proof. \( H: x \quad 18;Sx;x \)

t20. \( Sx - x = S0 \)

t21. \( Sx - x \neq 0 \)
Proof. \( H: x \quad 20;x \quad 3;0 \)

t22b. \( 0 \neq 0 \rightarrow 0 = SP0 \)
Proof. \( H \quad 5;0 \)

t22i. \( [x \neq 0 \rightarrow x = SPx] \rightarrow [Sx \neq 0 \rightarrow Sx = SPSx] \)
Proof. \( H: x \quad 16;x \)

t22. \( x \neq 0 \rightarrow x = SPx \)

t23b. \( y - 0 \neq 0 \rightarrow 0 + (y - 0) = y \)
Proof. \( H: y \quad 17;y \quad 13;y \quad 12;y \)

t23i. \( [y - x \neq 0 \rightarrow x + (y - x) = y] \rightarrow [y - Sx \neq 0 \rightarrow Sx + (y - Sx) = y] \)
Proof. \( H: y; x \quad 17;y;x \quad 16;y-x \quad 14;x;P(y-x) \quad 22;y-x \)

t23. \( y - x \neq 0 \rightarrow x + (y - x) = y \)

t24. \( y - x = S0 \rightarrow Sx = y \)
Proof. \( H: y; x \quad 3;0 \quad 23;y;x \quad 12;x:0 \)

t25. \( x \neq 0 \& Px = 0 \rightarrow x = S0 \)
Proof. \( H: x \quad 22;x \)

t26. \( y - x \neq 0 \& y - Sx = 0 \rightarrow y - x = S0 \)
Proof. \( H: y; x \quad 17;y;x \quad 25;y-x \)
t27. \( y - x \neq 0 \rightarrow Sx = y \lor y - Sx \neq 0 \)

*Proof.* H: \( y : x \ 26 ; y : x \ 24 ; y : x \)

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t28b. \( x - 0 \neq 0 \rightarrow Sx - 0 \neq 0 \)

*Proof.* H: \( x : Sx \ 3 : x \)
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t28i. \( [x - y \neq 0 \rightarrow Sx - y \neq 0] \rightarrow [x - Sy \neq 0 \rightarrow Sx - Sy \neq 0] \)

*Proof.* H: \( x : y \ 18 ; x : y \ 17 ; x : y \ 16 \)

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t28. \( x - y \neq 0 \rightarrow Sx - y \neq 0 \)
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t29b. \( 0 = y \lor y - 0 \neq 0 \lor 0 - y \neq 0 \)

*Proof.* H: \( y : 17 ; y \)

t29i. \( [x = y \lor y - x \neq 0 \lor x - y \neq 0] \rightarrow [Sx = y \lor y - Sx \neq 0 \lor Sx - y \neq 0] \)

*Proof.* H: \( x : y \ 21 ; x \ 27 ; y : x \ 28 ; x : y \)

t29. \( x = y \lor y - x \neq 0 \lor x - y \neq 0 \)

t30. \( x - y = 0 \land y - x = 0 \rightarrow x = y \)

*Proof.* H: \( x : y \ 29 ; x : y \)

e31. \( Eq(x, y) = (x - y) + (y - x) \)

Here at last is the Hilbert-Bernays construction.

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t32. \( Eq(x, y) = 0 \leftrightarrow x = y \)**

*Proof.* H: \( x : y \ 31 ; x : y \ 15 ; x - y ; y - x \ 30 ; x : y \ 19 ; x \ 12 ; 0 \ ? Eq(x, y) \neq 0 \)
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The ? indicates the introduction of a claim.

#3. Using the case function symbol \( C \) such that \( C(x, y, z) = y \) if \( x \) is 0 and is \( z \) otherwise, form the characteristic term \( \chi A \) so that \( \vdash \chi A = 0 \leftrightarrow A \). Construct the formal system \( \chi PRA \), equivalent to PRA, whose only symbols are variables and PR function symbols: the logical connectives and \( = \) are eliminated. But continue working in PRA.

r33. \( C(0, y, z) = y \land C(Sx, y, z) = z \)

t34. \( x \neq 0 \rightarrow C(x, y, z) = z \)

*Proof.* H: \( x : y ; z \ 22 ; x \ 33 ; y ; z ; P x \)

t35. \( C(x, y, z) \neq z \rightarrow x = 0 \land C(x, y, z) = y \)

*Proof.* H: \( x : y ; z \ 34 ; x : y ; z \ 33 ; y ; z \)

t36. \( C(x, 0, S0) = 0 \leftrightarrow x = 0 \)

*Proof.* H: \( x \ 22 ; x \ 33 ; 0 ; S0 ; P x \ 3 ; 0 \ ? x \neq 0 \)

t37. \( C(x, S0, 0) = 0 \leftrightarrow x \neq 0 \)

*Proof.* H: \( x \ 22 ; x \ 33 ; S0 ; 0 ; P x \ 3 ; 0 \ ? x = 0 \)

t38. \( C(x, 0, S0) = 0 \lor C(x, 0, S0) = S0 \)

*Proof.* H: \( x \ 22 ; x \ 33 ; 0 ; S0 ; P x \)
t39. \( C(x, S0, 0) = 0 \lor C(x, S0, 0) = S0 \)

Proof. \( H: x \ 22; x \ 33; S0; 0; P\)

t40. \( C(x, 0, 0) = 0 \)

Proof. \( H: x \ 22; x \ 33; 0; 0; P\)

t41a. \( C(x, 0, C(y, 0, S0)) = 0 \rightarrow x = 0 \lor y = 0 \)

Proof. \( H: x: y \ 36; y \ 22; x \ 33; 0; C(y, 0, S0); P\)

t41b. \( x = 0 \lor y = 0 \rightarrow C(x, 0, C(y, 0, S0)) = 0 \)

Proof. \( H: x: y \ 36; y \ 22; x \ 33; 0; C(y, 0, S0); P\)

t41. \( C(x, 0, C(y, 0, S0)) = 0 \leftrightarrow x = 0 \lor y = 0 \)

Proof. Tautological consequence of 41a and 41b.

Think of 0 as true and S0 as false.

e42. \( x \dot{=} y = C(Eq(x, y), 0, S0) \)
e43. \( \dot{x} = C(x, S0, 0) \)
e44. \( x \dot{\lor} y = C(x, 0, C(y, 0, S0)) \)
t45. \( x \dot{=} y = 0 \lor x \dot{=} y = S0 \)

Proof. \( H: x: y \ 42; x: y \ 38; Eq(x, y) \)
t46. \( \dot{x} = 0 \lor \dot{x} = S0 \)

Proof. \( H: x \ 43; x \ 39; x \)
t47. \( x \dot{\lor} y = 0 \lor x \dot{\lor} y = S0 \)

Proof. \( H: x: y \ 44; x: y \ 38; y \ 40; x \ 38; x \)
e48. \( x \dot{\land} y = \dot{\dot{x}} \dot{\lor} \dot{\dot{y}} \)
e49. \( x \dot{\rightarrow} y = \dot{x} \dot{\lor} y \)
e50. \( x \dot{\leftrightarrow} y = (\dot{x} \dot{\lor} y) \dot{\land} (x \dot{\lor} \dot{y}) \)

To each A associate a term \( \chi_A \), the characteristic term of A, recursively as follows.

(1) \( \chi[a = b] \) is \( a \dot{=} b \)
(2) \( \chi[\neg B] \) is \( \dot{\neg} \chi B \)
(3) \( \chi[B \lor C] \) is \( \chi B \dot{\lor} \chi C \)

That is, \( \chi A \) is obtained by replacing each \( = \) by \( \dot{=} \), each \( \neg \) by \( \dot{\neg} \), and each \( \lor \) by \( \dot{\lor} \). From this it follows that

(4) \( [\chi A]_{\vec{x}(\vec{a})} \) is \( \chi[A_{\vec{x}(\vec{a})}] \)

If \( \ell \) is a decorated roman letter occurring in an expression schema v, then \( v, \ell/u \) is the expression or expression schema obtained by replacing each occurrence of \( \ell \) in \( v \) by \( u \).
PROPOSITION 2. The following are theorem schemata of PRA.

T51. $\chi A = 0 \iff A$

T52. $\chi A = 0 \lor \chi A = S0$

Proof. We have 51, $A/a = b$ by (1) and 42; if $A/B$ then 51, $A/\neg B$ by (2) and 43; $\chi B$ and 37; $\chi B$. If 51, $A/B$ and 51, $A/C$ then 51, $A/B \lor C$ by (3) and 47; $\chi B; \chi C$ and 41; $\chi B; \chi C$. By metamathematical induction on the formation of formulas, each $\chi A = 0 \iff A$ is a theorem of PRA. □

T53. $\chi A = S0 \iff \neg A$

Proof. Tautological consequence of 51 and 52 and 3; 0.

T54. $[A \iff B] \iff \chi A = \chi B$

Proof. Tautological consequence of 51 and 51, $A/B$ and 52 and 52, $A/B$ and 3; 0.

Now reformulate PRA as a formal system $\chi$PRA with a simpler data structure. The symbols of $\chi$PRA are the variables and the PR function symbols. Terms are as before. A $\chi$-equation is a term of the form $a \equiv b$. The formulas of $\chi$PRA, called $\chi$-formulas, are defined recursively as follows. A $\chi$-equation is a $\chi$-formula; if $b$ and $c$ are $\chi$-formulas, so are $\neg b$ and $b \lor c$. Decorated $\alpha \beta \gamma \delta$ stand for $\chi$-formulas. Note that $\chi$ is bijective from formulas of PRA onto formulas of $\chi$PRA; each $\alpha$ is $\chi A$ for a unique $A$, $\chi^{-1}\alpha$.

Think of the $\chi$-formula $\alpha$ as asserting that the term $\alpha$ is equal to 0. The axioms of $\chi$PRA are the characteristic terms of the axioms of PRA; the rules of inference of $\chi$PRA are formed from the rules of inference of PRA by replacing each premise and conclusion by its characteristic term.

Explicitly, the axioms and rules of inference of $\chi$PRA are as follows, where in $A\chi1$ and $A\chi2$, $f$ is the function symbol defined by $A1$ and $A2$.

A1. $f(x, y) \equiv a$
A2. $f(x, Sy) \equiv b_z(f(x, y))$
A3. $\neg Sx \equiv 0$
A4. $Sx \equiv Sy \rightarrow x \equiv y$
A5. $x \equiv x$
A6. $x \equiv y \rightarrow y \equiv x$
A7. $x \equiv y \& \alpha_x(x) \rightarrow \alpha_x(y)$
A8. $\alpha \lor \alpha \rightarrow \alpha$
A9. $\alpha \rightarrow \alpha \lor \beta$
A10. $\alpha \lor \beta \rightarrow \beta \lor \alpha$
A11. $(\beta \rightarrow \gamma) \rightarrow (\alpha \lor \beta \rightarrow \alpha \lor \gamma)$

$\chi$-instance: from $\alpha$ infer an instance of $\alpha$
$\chi$-modus ponens: from $\alpha$ and $\alpha \rightarrow \beta$ infer $\beta$
$\chi$-induction: from $\alpha_x(0)$ and $\alpha_x(x') \rightarrow \alpha_x(Sx')$ infer $\alpha_x(x)$
A proof in $\chi$PRA is a $\chi$-proof. Decorated $\sigma$ stands for a finite sequence of $\chi$-formulas, $\sigma \vdash^x \alpha$ asserts that $\sigma$ is a $\chi$-proof of $\alpha$, and $\vdash^x \alpha$ asserts that $\alpha$ is a theorem of $\chi$PRA.

**Proposition 3.** The following are equivalent: $\vdash A$ and $\vdash \chi A = 0$ and $\vdash^x \chi A$.

**Proof.** The first two are equivalent by T51. Let $\pi \vdash A$. Then $\chi \circ \pi \vdash^x \chi A$—for if $B$ in $\pi$ is an axiom, so is $\chi B$, and if $B$ is inferred by a rule of inference, then $\chi B$ is inferred by the corresponding rule. Conversely, if $\sigma \vdash^x \chi A$, let $\pi$ consist of all $B$ of the form $\chi^{-1} \beta = 0$ for $\beta$ in $\sigma$. If $\beta$ is an axiom, then $\chi^{-1} \beta$ is an axiom $D$ of PRA, so $B$ is $\chi D = 0$, which is a theorem by T51, $A/D$. If $\beta$ is inferred by a rule of inference, then $B$ is inferred by the corresponding rule. Hence $\pi$ is a proof with citation of theorems from the theorem schema T51, so $\vdash A$. \qed

#4. Establish primitive recursion by cases, though it will not be used until much later.

The following Proposition expresses the familiar "if, else if, . . . , else if, else" pattern for cases.

**Proposition 4.** Let $d = C\chi A_1 b_1 C\chi A_2 b_2 \ldots C\chi A_\nu b_\nu c_\nu$ and for $1 \leq \mu \leq \nu$ let $B_\mu$ be $\neg A_1 \& \cdots \& \neg A_\mu$. Then

$$\vdash (A_1 \to d = b_1) \& (B_1 \& A_2 \to d = b_2) \& \cdots \& (B_{\nu-1} \& A_\nu \to d = b_\nu) \& (B_\nu \to d = c_\nu) \tag{5}$$

**Proof.** First let $\nu = 1$. Then $d$ is $C\chi A_1 b_1 c_1$. We have $\chi A_1 = 0 \iff A_1$ by T51, $A/A_1$, so $A_1 \to d = b_1$ by 33:b1;c1. We have $\chi A_1 = S0 \iff \neg A_1$ by T53, $A/A_1$, so $\neg A_1 \to d = c_1$ by 33:b1;c1:0, proving the result for $\nu = 1$. Now assume as metamathematical induction hypothesis that the result holds for $\nu - 1$ and let $c'_1$ be $C\chi A_2 b_2 \ldots C\chi A_\nu b_\nu c_\nu$. By the case $\nu = 1$, $A_1 \to d = b_1$ and $\neg A_1 \to d = c'_1$, so (5) holds by the induction hypothesis. \qed

If $f$ is defined by the primitive recursion $f(\bar{x}, y) = a \& f(\bar{x}, Sy) = d$ (where $d$ is as in the Proposition) then we say that (5) holds by primitive recursion by cases.

5. Bounded quantifiers

#5. Given a formula $A$ and a variable $x$, construct the PR function symbol $\mu_A$ so that $\mu_A(x)$ (with the other variables in $A$ not indicated in the notation) finds the first $x'$, if any, such that $A_x(x')$ holds.

Introduce the PR predicate symbols $\leq$ (less than) and $<$ (strictly less than).

| d55  | $x \leq y$ | $x - y = 0$ |
| d56  | $x < y$ | $x \leq y \& x \neq y$ |
| t57  | $x \leq 0$ | $x = 0$ |

**Proof.** $H; x \quad 55 \rightarrow; x; 0 \quad 17; x$

If $d\xi$ is $p(\bar{x}) \leftrightarrow D$, then $\xi \rightarrow^x$ is $\neg p(\bar{x}) \lor D$ (which is $p(\bar{x}) \rightarrow D$) and $\xi \leftarrow^x$ is $p(\bar{x}) \lor \neg D$ (which is tautologically equivalent to $D \rightarrow p(\bar{x})$).
t58. $0 \leq x$
   \textit{Proof.} $H: x \ 55^<:0;x \ 23;0;x \ 15;x:0-x$

t59. $\neg Sx \leq x$
   \textit{Proof.} $H: x \ 55^>:Sx;x \ 21;x$

t60. $x \leq x$
   \textit{Proof.} $H: x \ 55^<:x;x \ 19;x$

t61. $x \leq y \rightarrow x < y \lor x = y$
   \textit{Proof.} $H:x:y \ 56^<:x;y$

t62. $x < y \lor x = y \rightarrow x \leq y$
   \textit{Proof.} $H:x:y \ 56^>:x;y \ 60;x$

t63. $x \leq Sx$
   \textit{Proof.} $H: x \ 17;x;x \ 19;x \ 16 \ 55^<:x:Sx$

t64b. $P0 \leq 0$
   \textit{Proof.} $H \ 16 \ 60;0$

t64i. $Px \leq x \rightarrow PSx \leq Sx$
   \textit{Proof.} $H: x \ 16;x \ 63;x$

t64. $Px \leq x$

t65. $x < y \rightarrow y - x \neq 0$
   \textit{Proof.} $H:x:y \ 56^>:x:y \ 55^>:x:y \ 29;x:y$

t66. $x < y \rightarrow y = x + (y - x)$
   \textit{Proof.} $H:x:y \ 65;x:y \ 23;y:x$

t67. $x = y \rightarrow y = x + (y - x)$
   \textit{Proof.} $H:x:y \ 19;y \ 12;x$

t68. $x \leq y \rightarrow y = x + (y - x)$
   \textit{Proof.} $H:x:y \ 56^<:x:y \ 67;x:y \ 66;x:y$

t69b. $x - (y + 0) = (x - y) - 0$
   \textit{Proof.} $H:x:y \ 12;y \ 17;x-y$

t69i. $[x - (y + z) = (x - y) - z] \rightarrow [x - (y + Sz) = (x - y) - Sz]$
   \textit{Proof.} $H:x:y:z \ 12;y:z \ 17;x:y+z \ 17;x-y;z$

t69. $x - (y + z) = (x - y) - z$

t70b. $0 - 0 = 0$
   \textit{Proof.} $H \ 19;0$
t70i. \( 0 - x = 0 \) \( \rightarrow \) \( 0 - Sx = 0 \)

Proof. \( H: x \) 17:0; \( x \) 16

t70. \( 0 - x = 0 \)

t71. \( x \leq x + y \)

Proof. \( H: x:y \) 69; \( x:y \) 19; \( x \) 70; \( y \) 55; \( x:x+y \)

t72b. \( x + (y + 0) = (x + y) + 0 \)

Proof. \( H: x:y \) 12; \( y \) 12; \( x+y \)

t72i. \( x + (y + z) = (x + y) + z \) \( \rightarrow \) \( x + (y + Sz) = (x + y) + Sz \)

Proof. \( H: x:y:z \) 12; \( y:z \) 12; \( x:y+z \) 12; \( x+y:z \) 4; \( x+(y+z):(x+y)+z \)

t72. \( x + (y + z) = (x + y) + z \)

t73. \( x \leq y \) \& \( y \leq x \) \( \rightarrow \) \( x \leq z \)

Proof. \( H: x:y:z \) 68; \( x:y \) 68; \( y:z \) 72; \( x:y-x:z-y \) 71; \( x:(y-x)+(z-y) \)

t74. \( x < y \) \( \rightarrow \) \( y - x \neq 0 \)

Proof. \( H: x:y \) 56; \( x:y \) 68; \( x:y \) 12; \( x \)

t75. \( y - x \neq 0 \) \( \rightarrow \) \( x < y \)

Proof. \( H: y:x \) 23; \( y:x \) 71; \( x:y-x \) 56; \( x:y \) 19; \( x \)

t76. \( x \leq y \) \& \( y \leq x \) \( \rightarrow \) \( x = y \)

Proof. \( H: x:y \) 55; \( x:y \) 55; \( y:x \) 30; \( x:y \)

t77. \( x < y \) \( \lor \) \( x = y \) \( \lor \) \( y < x \)

Proof. \( H: x:y \) 75; \( x:y \) 75; \( y:x \) 30; \( x:y \)

t78. \( x < y \) \( \rightarrow \) \( \neg y \leq x \)

Proof. \( H: x:y \) 56; \( x:y \) 76; \( x:y \)

t79. \( \neg y \leq x \) \( \rightarrow \) \( x < y \)

Proof. \( H: y:x \) 77; \( x:y \) 56; \( y:x \) 60; \( x \)

t80. \( \neg [x < y \) \& \( y \leq x] \)

Proof. \( H: x:y \) 78; \( x:y \)

t81. \( Sx \leq y \) \( \rightarrow \) \( x < y \)

Proof. \( H: x:y \) 56; \( x:y \) 63; \( x \) 73; \( Sx:y \) 55; \( Sx:x \) 21; \( x \)

Let \( \bar{y} \) be the distinct variables of \( A \) other than \( x \) in the order of first occurrence, and let \( \mu_A(a) \) abbreviate \( \mu_A(\bar{y}, a) \). Define the PR function symbol \( \mu_A \) by

R82. \( \mu_A(0) = C([A_x(0)], 0, S0) \) \&
\( \mu_A(Sx) = C([\mu_A(x) \leq x], \mu_A(x), C([A_x(Sx)], Sx, SSx)) \)

Remark that if \( x_1 \) is any variable other than those in \( \bar{y} \) and \( A_1 \) is \( A_x(x_1) \), then \( \mu_{A_1} \) is the same as \( \mu_A \). We have \( \mu_A(x) = Sx \) until an \( x' \) (if any) is found such that \( A_x(x') \) holds,
after which it remains \( x' \) forever, as we now demonstrate (with implicit uses of equality axioms and symmetry).

\[
\text{T83b. } \mu_A(0) \leq 0 \rightarrow A_x(\mu_A(0))
\]

\textit{Proof.} Suppose not. Then
\begin{align*}
.1 & \quad \mu_A(0) \leq 0 \\
.2 & \quad \neg A_x(\mu_A(0))
\end{align*}

By .1 and 57; \( \mu_A(0) \),
\begin{align*}
.3 & \quad \mu_A(0) = 0
\end{align*}

By .3 and 82,
\begin{align*}
.4 & \quad C(\chi[A_x(0)], 0, S0) = 0
\end{align*}

By .4 and 36; \( \chi[A_x(0)] \),
\begin{align*}
.5 & \quad \chi[A_x(0)] = 0
\end{align*}

By .5 and 51, \( A/A_x(0) \),
\begin{align*}
.6 & \quad A_x(0)
\end{align*}

By .2 and .3,
\begin{align*}
.7 & \quad \neg A_x(0)
\end{align*}

QEA by .6 and .7.

\[
\text{T83i. } [\mu_A(x) \leq x \rightarrow A_x(\mu_A(x))] \rightarrow [\mu_A(Sx) \leq Sx \rightarrow A_x(\mu_A(Sx))]
\]

\textit{Proof.} Suppose not. Then
\begin{align*}
.1 & \quad \mu_A(x) \leq x \rightarrow A_x(\mu_A(x)) \\
.2 & \quad \mu_A(Sx) \leq Sx \\
.3 & \quad \neg A_x(\mu_A(Sx))
\end{align*}

Claim: \( \neg \mu_A(x) \leq x \). Suppose not. Then
\begin{align*}
.4 & \quad \mu_A(x) \leq x
\end{align*}

By .4 and 51, \( A/\mu_A(x) \leq x \),
\begin{align*}
.5 & \quad \chi[\mu_A(x)] = 0
\end{align*}

By .4 and 82,
\begin{align*}
.6 & \quad \mu_A(Sx) = C(0, \mu_A(x), C(\chi[A_x(Sx)], Sx, SSx))
\end{align*}

By .6 and 33; \( \mu_A(x) \),
\begin{align*}
.7 & \quad \mu_A(Sx) = \mu_A(x)
\end{align*}

By .1 and .4,
\begin{align*}
.8 & \quad A_x(\mu_A(x))
\end{align*}

By .7 and .8,
\begin{align*}
.9 & \quad A_x(\mu_A(Sx))
\end{align*}

The claim is proved by .3 and .9, and .4–.9 will not be used again.
\begin{align*}
.10 & \quad \neg \mu_A(x) \leq x
\end{align*}

By .10 and 53, \( A/\mu_A(x) \leq x \),
\begin{align*}
.11 & \quad \chi[\mu_A(x) \leq x] = S0
\end{align*}

By .11 and 82,
\begin{align*}
.12 & \quad \mu_A(Sx) = C(S0, \mu_A(x), C(\chi[A_x(Sx)], Sx, SSx))
\end{align*}

By .12 and 33; \( \mu_A(x); C(\chi[A_x(Sx)]), Sx, SSx), 0 \),
\begin{align*}
.13 & \quad \mu_A(Sx) = C(\chi[A_x(Sx)], Sx, SSx)
\end{align*}

By .2 and 59; \( Sx \),
\begin{align*}
.14 & \quad \mu_A(Sx) \neq SSx
\end{align*}
By .14 and 35; \( \chi [A_x(S_x); S_x; SS_x] \),
\[ \chi [A_x(S_x)] = 0 \& C(\chi [A_x(S_x)], S_x, SS_x) = S_x \]
By .15 and 51, \( A_x/(A_x(S_x)) \),
\[ \mu_A(S_x) = S_x \]
By .13 and .15,
\[ \mu_A(S_x) = S_x \]
By .16 and .17,
\[ \mu_A(\mu_A(S_x)) \]
QEA by .3 and .18.

T83. \( \mu_A(x) \leq x \rightarrow A_x(\mu_A(x)) \)

T84b. \( A_x(0) \rightarrow \mu_A(0) \leq 0 \)

**Proof.** Suppose not. Then
\[ \mu_A(0) \leq 0 \]
By .1 and 49, \( A_x(0) \),
\[ \chi [A_x(0)] = 0 \]
By .3 and 82,
\[ \mu_A(0) = C(0, S_0, SS_0) \]
By .4 and 40:0,
\[ \mu_A(0) = 0 \]
By .5 and 60:0,
\[ \mu_A(0) \leq 0 \]
QEA by .2 and .7.

T84i. \( [A_x(x) \rightarrow \mu_A(x) \leq x] \rightarrow [A_x(S_x) \rightarrow \mu_A(S_x) \leq S_x] \)

**Proof.** Suppose not. Then (the induction hypothesis is not needed in this proof)
\[ A_x(S_x) \]
\[ \mu_A(S_x) \leq S_x \]
By .1 and 51, \( A_x/(A_x(S_x)) \),
\[ \chi [A_x(S_x)] = 0 \]
Claim: \( \mu_A(x) \leq x \). Suppose not. Then
\[ \mu_A(x) \leq x \]
By .4 and 53, \( A/\mu_A(x) \leq x \),
\[ \chi [\mu_A(x)] = S_0 \]
By .5 and 82,
\[ \mu_A(S_x) = C(S_0, \mu_A(x), C(\chi [A_x(S_x)], S_x, SS_x)) \]
By .6 and 33; \( \mu_A(x); C(\chi [A_x(S_x)], S_x, SS_x); 0 \),
\[ \mu_A(S_x) = C(\chi [A_x(S_x)], S_x, SS_x) \]
By .3 and .7,
\[ \mu_A(S_x) = C(0, S_x, SS_x) \]
By .8 and 33; \( S_x; SS_x \),
\[ \mu_A(S_x) = S_x \]
By .9 and 60; \( S_x \),
\[ \mu_A(S_x) \leq S_x \]
The claim is proved by .2 and .10, and .4–.10 will not be used again.
\[ \mu_A(x) \leq x \]

By .11 and 51, \( A/\mu_A(x) \leq x \),

\[ \chi[\mu_A(x) \leq x] = 0 \]

By .12 and 82,

\[ \mu_A(Sx) = C(0, \mu_A(x), C(\chi[A_x(Sx)], Sx, SSx)) \]

By .13 and 33; \( \mu_A(x); C(\chi[A_x(Sx)], Sx, SSx) \),

\[ \mu_A(Sx) = \mu_A(x) \]

By .14 and .11,

\[ \mu_A(Sx) \leq x \]

By .15 and 63;x and 73; \( \mu_A(x); x; Sx \),

\[ \mu_A(Sx) \leq Sx \]

QEA by .2 and .16.

T84. \( A_x(x) \rightarrow \mu_A(x) \leq x \)

T85. \( A_x(x) \rightarrow A_x(\mu_A(x)) \) \& \( \mu_A(x) \leq x \)

**Proof.** By 84 and 83.

T86b. \( \mu_A(x) \leq x \rightarrow \mu_A(x + 0) = \mu_A(x) \)

**Proof.** By 12;x and 5; \( \mu_A(x) \).

T86i. \( [\mu_A(x) \leq x \rightarrow \mu_A(x + w) = \mu_A(x)] \rightarrow [\mu_A(x) \leq x \rightarrow \mu_A(x + Sw) = \mu_A(x)] \)

**Proof.** Suppose not. Then

\[ \mu_A(x) \leq x \rightarrow \mu_A(x + w) = \mu_A(x) \]

\[ \mu_A(x) \leq x \]

\[ \mu_A(x + Sw) \neq \mu_A(x) \]

By .2 and .1,

\[ \mu_A(x + w) = \mu_A(x) \]

By 12;x;w,

\[ x + Sw = S(x + w) \]

By .4 and .2,

\[ \mu_A(x + w) \leq x \]

By .6 and 71;x;w and 73; \( \mu_A(x + w); x + w \),

\[ \mu_A(x + w) \leq x + w \]

By .7 and 51, \( A/\mu_A(x + w) \leq x + w, \)

\[ \chi[\mu_A(x + w) \leq x + w] = 0 \]

By .8 and 82;x + w,

\[ \mu_A(S(x + w)) = C(0, \mu_A(x + w), C(\chi[A_x(S(x + w))], S(x + w), SS(x + w))) \]

By .9 and .5 and 33; \( \mu_A(x + w); C(\chi[A_x(S(x + w))], S(x + w), SS(x + w)) \),

\[ \mu_A(x + Sw) = \mu_A(x + w) \]

By .10 and .4,

\[ \mu_A(x + Sw) = \mu_A(x) \]

QEA by .3 and .11.

T86. \( \mu_A(x) \leq x \rightarrow \mu_A(x + w) = \mu_A(x) \)

T87. \( \mu_A(x) \leq x \) \& \( x \leq z \rightarrow \mu_A(z) = \mu_A(x) \)
Proof. Suppose not. Then
.1 $\mu_A(x) \leq x$
.2 $x \leq z$
.3 $\mu_A(z) \neq \mu_A(x)$
By .2 and 68:x;z,
.4 $z = x + (z - x)$
By .1 and .4 and 86:x;z − x,
.5 $\mu_A(z) = \mu_A(x)$
QEA by .3 and .5.

#6. Introduce bounded quantifiers $\exists x \leq b A$ and $\forall x \leq b A$, where $x$ does not occur in $b$, as instances of $A$, using $\mu_A$.

If $x$ does occur in $b$, let $\exists x \leq b A$ abbreviate $A_x(\mu_A(b))$, and let $\forall x \leq b A$ abbreviate $A_x(\mu_{\neg A}(b))$. Then $\forall x \leq b A$ is tautologically equivalent to $\neg \exists x \leq b \neg A$, since the latter is the double negation of the former. Call $\exists x \leq b$ and $\forall x \leq b$ bounded quantifiers; they occur only in abbreviations, and $x$ does not occur in $\exists x \leq b A$ or $\forall x \leq b A$. Each is an instance of $A$.

T88. $x \leq b \& A_x(x) \rightarrow \exists x \leq b A$

Proof. Suppose not. Then
.1 $x \leq b$
.2 $A_x(x)$
.3 $\neg A_x(\mu_A(b))$
By .2 and 85,
.4 $A_x(\mu_A(x)) \& \mu_A(x) \leq x$
By .4 and .1 and 87;x;b,
.5 $\mu_A(b) = \mu_A(x)$
By .5 and .4,
.6 $A_x(\mu_A(b))$
QEA by .3 and .6.

For the converse we have a specific number, $\mu_A(b)$, that is less than $b$ and satisfies $A$.

T89. $\exists x \leq b A \rightarrow \mu_A(b) \leq b \& A_x(\mu_A(b))$  

Proof. By 85;\mu_{\neg A}(b) (recalling that $\exists x \leq b A$ is $A_x(\mu_A(b))$).

T90. $\forall x \leq b A \rightarrow [x \leq b \rightarrow A_x(x)]$  

Proof. Tautologically equivalent to 88, $A / \neg A$.

T91. $\exists x \leq c A \& c \leq b \rightarrow \exists x \leq b A$

Proof. Suppose not. Then
.1 $A_x(\mu_A(c))$
.2 $c \leq b$
.3 $\neg A_x(\mu_A(b))$
By 89, $b/c$,
.4 $\mu_A(c) \leq c \& A_x(\mu_A(b))$
By 87;\mu_{\neg A}(c)c;b and .2,
.5 $\mu_A(b) = \mu_A(c)$
QEA by .1 and .3 and .5.

T92. $\forall x \leq b A \& c \leq b \rightarrow \forall x \leq c A$

Proof. Tautologically equivalent to 91, $A \rightarrow \neg A$.

#7. Replace induction as a rule of inference by an axiom schema, and use this to construct formal systems $PRA^*$ and $\chi PRA^*$, equivalent to $PRA$, in which the only rule of inference is, respectively, modus ponens or $\chi$-modus ponens, and such that variable-free theorems have variable-free proofs. But continue working in $PRA$; $\chi PRA^*$ will be arithmetized later.

T93b. $A_x(0) \& \forall x' \leq Px[A_x(x') \rightarrow A_x(Sx')] \rightarrow A_x(0)$

Proof. Tautology.

T93i. $\{A_x(0) \& \forall x' \leq Px[A_x(x') \rightarrow A_x(Sx')] \rightarrow A_x(x)\} \rightarrow \{A_x(0) \& \forall x' \leq Psx[A_x(x') \rightarrow A_x(Sx')] \rightarrow A_x(Sx)\}$

Proof. Suppose not. Then

1. $A_x(0) \& \forall x' \leq Px[A_x(x') \rightarrow A_x(Sx')] \rightarrow A_x(x)$
2. $A_x(0)$
3. $\forall x' \leq Psx[A_x(x) \rightarrow A_x(Sx')]$
4. $\neg A_x(Sx)$

By .2 and .1,
5. $\forall x' \leq Px[A_x(x') \rightarrow A_x(Sx')] \rightarrow A_x(x)$

By 16:x,
6. $Psx = x$

By .6 and .3,
7. $\forall x' \leq x[A_x(x') \rightarrow A_x(Sx')]$

By 60:x,
8. $x \leq x$

By .8 and .7 and 90, $x/x'$, $b/x$, $A/A_x(x) \rightarrow A_x(Sx)$,
9. $A_x(x) \rightarrow A_x(Sx)$

By 64:x,
10. $Px \leq x$

By .10 and .7 and 90, $x/x'$, $b/Px, A/A_x(x) \rightarrow A_x(Sx)$,
11. $\forall x' \leq Px[A_x(x') \rightarrow A_x(Sx')]$

By .11 and .5,
12. $A_x(x)$

By .12 and .9,
13. $A_x(Sx)$

QEA by .4 and .13.

T93. $A_x(0) \& \forall x' \leq Px[A_x(x') \rightarrow A_x(Sx')] \rightarrow A_x(x)$

Now simplify $PRA$ even further, via formal theories $PRA^*$ and $\chi PRA^*$. The symbols, terms, and formulas of $PRA^*$ are those of $PRA$; its axioms are all instances of axioms of $PRA$ and of T93; its only rule of inference is modus ponens. The symbols, terms, and formulas of $\chi PRA^*$ are those of $\chi PRA$; its axioms are the characteristic terms of the axioms of $PRA^*$; its only rule of inference is $\chi$-modus ponens. Proofs and theorems of these two systems are indicated by $\vdash^*$ and $\vdash^{\chi*}$.

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PROPOSITION 5. (i) $\vdash A$ if and only if $\vdash^* A$.
(ii) A variable-free theorem of PRA* has a variable-free proof.
(iii) $\vdash^* A$ if and only if $\vdash^{x*}\chi A$.
(iv) A variable-free theorem of $\chi PRA^*$ has a variable-free proof.
(v) A variable-free theorem of $\chi PRA^*$ has a variable-free proof.

Proof. For (i), let $\pi \vdash A$ and let $B_x(x)$ be the first formula in $\pi$ that is inferred by induction, from $B_x(0)$ and $B_x(x') \to B_x(Sx')$. Then we have proofs of these two premises without using induction. By an instance of the second premise we have $\forall x \leq Px[B_x(x') \to B_x(Sx')]$, so we have $B_x(0) & \forall x \leq Px[B_x(x') \to B_x(Sx')]$ by tautological consequence. By 93, $A/B$ we have $B_x(x)$. Proceeding in this way, by metamathematical induction on the number of inferences by induction in $\pi$, we have a proof $\pi_1$ of $A$ from the axioms of PRA* without using induction. Now consider the first formula $C_x(\vec{a})$ of $\pi_1$ that is inferred by instance, from $C$. Let $\pi_2$ the the part of $\pi_1$ strictly preceding $C_x(\vec{a})$. Let $D'$ be $D_x(\vec{a})$, and let $\pi_2'$ consist of all $D'$ for $D$ in $\pi_2$. Then $\pi_2\pi_2'C'$ is a proof from the axioms of PRA* of C' (i.e., $C_x(\vec{a})$) without using induction or instance, because if $D$ is an axiom of PRA* so is $D'$ (since an instance of an instance is an instance), and if $D$ is inferred from $D_1$ and $D_1 \to D$ by modus ponens, then $D'$ is inferred from $D_1'$ and $D_1' \to D'$ by modus ponens (since $[D_1 \to D]'$ is $D_1' \to D'$). (Don’t delete $\pi_2$, because later formulas in $\pi_1$ may be inferred by instance from a formula in it.) Proceeding in this way. by metamathematical induction on the number of inferences by instance in $\pi_1$, we obtain a proof of $A$ in PRA*. The converse direction of (i) is trivial, since the axioms of PRA* are theorems of PRA and the rule of inference of PRA* is a rule of inference of PRA.

For (ii), let $A$ be variable-free with $\pi \vdash^* A$. Let $D^o$ be the formula obtained by replacing all variables in $D$ by 0, and let $\pi^o$ consist of all $D^o$ for $D$ in $\pi$. Then $\pi^o$ is a variable-free proof in PRA* of $A$, because if $D$ is an axiom of PRA* so is $D^o$; if $D$ is inferred by modus ponens from $D_1$ and $D_1 \to D$, then $D^o$ is inferred by modus ponens from $D_1^o$ and $D_1^o \to D^o$; and $A^o$ is $A$.

For (iii), let $\pi \vdash^* A$. Then $\chi \circ \pi \vdash^{x*}\chi A$—for if $B$ in $\pi$ is an axiom, so is $\chi B$, and if $B$ is inferred by modus ponens, then $\chi B$ is inferred by $\chi$-modus ponens. Conversely, if $\sigma \vdash^{x*}\chi A$, let $\pi$ consist of all $B$ of the form $\chi^{-1}\beta = 0$ for $\beta$ in $\sigma$. If $\beta$ is an axiom, then $\chi^{-1}\beta$ is an axiom of PRA*, which is a theorem of PRA, so $B$ is $\chi D = 0$, which is a theorem of PRA by T51, A/D. If $\beta$ is inferred by $\chi$-modus ponens, then $B$ is inferred by modus ponens. Hence $\pi$ is a proof in PRA with citation of theorems, so $\vdash A$. Consequently, $\vdash^* A$ by (i).

For (iv), let $\chi A$ be a variable-free theorem of $\chi$-PRA. By (iii), $A$ is a variable-free theorem of PRA*, which by (ii) has a variable-free proof $\pi$ in PRA*. As shown in the proof of (iii), $\chi \circ \pi \vdash^{x*}\chi A$, and this proof is variable-free. □

#8. Construct the least number principle as a derived rule of inference.

If the recursion for $\mu_A$ finds an $x'$ such that $A_x(x')$ holds, then $x'$ is the least number for which $A_x(x')$ holds. This leads to the least number principle, a powerful form of induction.

T94. $A_x(x) \to \{y < \mu_A(x) \to \neg A_x(y)\}$

Proof. Suppose not. Then

$.1 A_x(x)$
Let $\forall x < b \ A$ (where $x$ does not occur in $b$) abbreviate $\forall x < b [x < b \rightarrow A]$. Then $\forall x < b A$ is an instance of $x < b \rightarrow A$. The following theorem schema follows from T94 by instance.

**T95.** $A_x(x) \rightarrow \forall y < \mu_A(x)[\neg A_x(y)]$

**T96.** $A_x(x) \rightarrow \mu_A(\mu_A(x)) = \mu_A(x)$

*Proof. If $A$ does not have a least counterexample, then $A$ holds. That is, if $\vdash \neg \{\neg A_x(z) \& \forall y < z [A_x(y)]\}$ then $\vdash A_x(z)$.*

**Proposition 6.** If $A$ does not have a least counterexample, then $A$ holds. That is, if $\vdash \neg \{\neg A_x(z) \& \forall y < z [A_x(y)]\}$ then $\vdash A_x(z)$.

*Proof. Suppose $\vdash \neg \{\neg A_x(z) \& \forall y < z [A_x(y)]\}$. Then we prove $A_x(z)$ as follows. Suppose not. Then*
.1 \ \neg A_x(x)

By .1 and 85, A/\neg A,

.2 \ \neg A(\mu_A(x))

By .2 and 95, A/\neg A, x/\mu_A(x),

.3 \ \forall y < \mu_A(\mu_A(x))[A_x(y)]

By .3 and .1 and 96, A/\neg A,

.4 \ \forall y < \mu_A(x)[A_x(y)]

By .0; \mu_A(x),

.5 \ \neg \{\neg A_x(\mu_A(x)) \& \forall y < \mu_A(x)[A_x(y)]\}

By .5 and .2,

.6 \ \forall y < \mu_A(x)[A_x(y)]

QEA by .4 and .6. \Box

The derived rule of inference, from \neg \{\neg A_x(z) \& \forall y < z[A_x(y)]\} infer A, is the least number principle.

#9. Construct definition of function symbols with uniqueness condition and bounded existence condition.

PROPOSITION 7. Let A contain no variables other than the distinct variables \vec{y} and x. The uniqueness condition (UC) is A_x(x) \& A_x(x') \rightarrow x = x', where x' is distinct from the \vec{y} and x. The existence condition (EC) is \exists x \leq b A, where b contains no variables other than those in \vec{y}. If UC and EC are theorems, let f(\vec{y}), where f is a new function symbol, abbreviate \mu_A(b). Then \vdash f(\vec{y}) = x \leftrightarrow A.

Proof. By definition, EC is \mu_A(b), and since it is a theorem, \vdash A_x(\mu_A(b))—i.e., \vdash A_x(f(\vec{y}))—by T89. Consequently, \vdash f(\vec{y}) = x \leftrightarrow A. The converse holds by UC. \Box

Such function symbols are defined function symbols; they occur only in abbreviations.

6. Strings

A string is a concatenation of bits. Identify the number x with the string consisting of the ones and zeros following the initial one in the binary representation of Sx. We implement this in PRA.

#10. Prove that every non-zero number Sx can be written uniquely as Qx + Rx where Qx is a power of two and Rx is strictly less than Qx.

t97b. \ 0 + 0 = 0

Proof. H 12;0

t97i. \ 0 + x = x \rightarrow 0 + Sx = Sx

Proof. H:x 12;0;x

t97. \ 0 + x = x

t98b. \ x + 0 = 0 + x

Proof. H:x 12;x 97;x
t98. \( x + y = y + x \)

**Proof.**

\( H:\{ x \colon y \quad 12; \colon x \quad 12; y \colon 14; y \colon x \} \)

Introduce multiplication. As usual, \( \cdot \) binds more tightly than +.

r99. \( x \cdot 0 = 0 \quad \& \quad x \cdot Sy = x + x \cdot y \)

t100. \( x \cdot 0 = 0 \quad \& \quad x \cdot Sy = x \cdot y + x \)

**Proof.**

\( H:\{ x \colon y \quad 99; x \colon y \quad 98; x \colon x \} \)

t101b. \( x \cdot (y + 0) = x \cdot y + x \cdot 0 \)

**Proof.**

\( H:\{ x \colon y \quad 12; y \colon 100; x \colon 12; x \colon y \} \)

t101i. \( x \cdot (y + z) = x \cdot y + x \cdot z \quad \rightarrow \quad x \cdot (y + Sz) = x \cdot y + x \cdot Sz \)

**Proof.**

\( H:\{ x \colon y \colon z \quad 12; y \colon z \quad 100; x \colon z \quad 100; x \colon y + z \quad 72; x \colon y \colon z \colon x \} \)

t101. \( x \cdot (y + z) = x \cdot y + x \cdot z \)

t102b. \( x \cdot (y \cdot 0) = (x \cdot y) \cdot 0 \)

**Proof.**

\( H:\{ x \colon y \quad 100; y \colon 100; x \colon 100; x \colon y \} \)

t102i. \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \rightarrow \quad x \cdot (y \cdot Sz) = (x \cdot y) \cdot Sz \)

**Proof.**

\( H:\{ x \colon y \colon z \quad 100; y \colon z \quad 100; x \colon y \colon z \quad 101; x \colon y \colon z \colon y \} \)

t102. \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \)

t103b. \( 0 \cdot 0 = 0 \)

**Proof.**

\( H \quad 100; 0 \)

t103i. \( 0 \cdot x = 0 \quad \rightarrow \quad 0 \cdot Sx = 0 \)

**Proof.**

\( H:\{ x \colon x \quad 100; 0 \colon x \quad 12; 0 \} \)

t103. \( 0 \cdot x = 0 \)

t104b. \( Sx \cdot 0 = x \cdot 0 + 0 \)

**Proof.**

\( H:\{ x \colon 100; Sx \colon 100; x \colon 12; 0 \} \)

t104i. \( Sx \cdot y = x \cdot y + y \quad \rightarrow \quad Sx \cdot Sy = x \cdot Sy + Sy \)

**Proof.**

\( H:\{ x \colon y \quad 100; x \colon y \quad 100; Sx \colon y \quad 72; x \colon y \colon Sy \quad 72; x \colon y \colon Sx \quad 14; y \colon x \quad 98; Sy \colon x \} \)

t104. \( Sx \cdot y = x \cdot y + y \)

t105b. \( x \cdot 0 = 0 \cdot x \)

**Proof.**

\( H:\{ x \colon x \colon 100; x \colon 103; x \} \)

t105i. \( x \cdot y = y \cdot x \quad \rightarrow \quad x \cdot Sy = Sy \cdot x \)

**Proof.**

\( H:\{ x \colon y \quad 100; x \colon y \quad 104; y \colon x \} \)

t105. \( x \cdot y = y \cdot x \)
t106. \((x + y) \cdot z = x \cdot z + y \cdot z\)

Proof. H: \(x:y:z\) 105;\(x+y:z\) 101;\(z:x:y\) 105;\(z:x\) 105;\(z:y\)

t107. \(x \leq y \rightarrow x \cdot z \leq y \cdot z\)

Proof. H: \(x:y:z\) 68;\(x:y\) 106;\(x:y-z\) 71;\(x:z:(y-x)\cdot z\)

t108. \(x < y \rightarrow x + (y - x) = y \quad \& \quad y - x \neq 0\)

Proof. H: \(x:y\) 68;\(x:y\) 56;\(z:x:y\) 12;\(x\)

t109. \(x < Sy \rightarrow x \leq y\)

Proof. H: \(x:y\) 108;\(x:Sy\) 22;\(Sy-x\) 12;\(x:P(Sy-x)\) 4;\(x+P(Sy-x)\);\(y\) 71;\(x:P(Sy-x)\)

t110. \(x \leq Sy \rightarrow x \leq y \lor x = Sy\)

Proof. H: \(x:y\) 61;\(x:Sy\) 109;\(x:y\)

t111. \(-[x < y \quad \& \quad y < Sx]\)

Proof. H: \(x:y\) 109;\(y:x\) 78;\(x:y\)

t112. \(x \leq y \rightarrow Sx \leq Sy\)

Proof. H: \(x:y\) 68;\(x:y\) 12;\(x:y-x\) 14;\(x:y-x\) 71;\(Sx:y-x\)

t113. \(x < y \rightarrow Sx < Sy\)

Proof. H: \(x:y\) 112;\(x:y\) 56;\(x:y\) 56;\(Sx:Sy\) 4;\(x:y\)

t114. \(x < y \rightarrow Sx \leq y\)

Proof. H: \(x:y\) 108;\(x:y\) 22;\(y-x\) 14;\(x:P(y-x)\) 71;\(Sx:P(y-x)\)

e115. 1 = S0

e116. 2 = SS0

t117. \(1 \cdot x = x\)

Proof. H: \(x\) 115 104;\(0:x\) 97;\(x\) 103;\(x\)

t118. \(2 \cdot x = x + x\)

Proof. H: \(x\) 116 115 104;\(1:x\) 117;\(x\)

t119b. \(y + 0 = z + 0 \rightarrow y = z\)

Proof. H: \(y:z\) 12;\(y\) 12;\(z\)

t119i. \([y + x = z + x \rightarrow y = z]\) \rightarrow \([y + Sx = z + Sx \rightarrow y = z]\)

Proof. H: \(y:x:z\) 12;\(y:x\) 12;\(z:x\) 4;\(y+x:z+x\)

t119. \(y + x = z + x \rightarrow y = z\)

t120. \(x + y = x + z \rightarrow y = z\)

Proof. H: \(x:y:z\) 98;\(x:y\) 98;\(x:z\) 119;\(y:x:z\)

t121. \(y \neq 0 \rightarrow x < x + y\)

Proof. H: \(y:x\) 71;\(x:y\) 56;\(x:x+y\) 12;\(x:0:y\) 12;\(x\)
t122. \( x \leq y \quad \& \quad y < z \rightarrow x < z \)
\textbf{Proof.} \( H : x ; y ; z \quad 56 \rightarrow ; y ; z \quad 73 ; x ; y ; z \quad 56 \leq ; x ; z \quad 80 ; y ; x \)

t123. \( x < y \rightarrow y \neq 0 \)
\textbf{Proof.} \( H : x ; y \quad 56 \rightarrow ; x ; y \quad 57 ; x \)

t124. \( x < y \rightarrow Sx < 2 \cdot y \)
\textbf{Proof.} \( H : x ; y \quad 118 ; y \quad 114 ; x ; y \quad 123 ; x ; y \quad 121 ; y ; y \quad 122 ; Sx ; y ; y + y \)

t125. \( x < Sx \)
\textbf{Proof.} \( H : x \quad 59 ; x \quad 79 ; Sx ; x \)

Introduce \textit{exponentiation}.

r126. \( x \uparrow 0 = 1 \quad \& \quad x \uparrow Sy = x \cdot (x \uparrow y) \)

t127b. \( 0 < 2 \uparrow 0 \)
\textbf{Proof.} \( H \quad 116 \quad 115 \quad 126 ; 2 \quad 125 ; 0 \)

t127i. \( x < 2 \uparrow x \rightarrow Sx < 2 \uparrow Sx \)
\textbf{Proof.} \( H : x \quad 126 ; 2 ; x \quad 124 ; x ; 2 \uparrow x \)

t127. \( x < 2 \uparrow x \)

t128. \( x \leq 2 \uparrow x \)
\textbf{Proof.} \( H : x \quad 127 ; x \quad 56 \rightarrow ; x ; 2 \uparrow x \)

d129. \( q \text{ is a power of two} \leftrightarrow \exists x \leq q [2 \uparrow x = q] \quad * \)

Although “is a power of two” contains English words, it is a formal predicate symbol, written in suffix notation. The negation of “\( q \text{ is a power of two} \)” is “\( \neg q \text{ is a power of two} \)”, whereas “\( q \text{ is not a power of two} \)” is not even an expression.

t130. \( 1 \text{ is a power of two} \)
\textbf{Proof.} \( H \quad 115 \quad 129 \leq ; 1 ; 0 \quad 128 ; 0 \quad 126 ; 2 \)

Here is the explanation of the notation 129<;1;0 in the proof. The formula 129< is
\( q \text{ is a power of two} \lor \neg \exists x \leq q [2 \uparrow x = q] \)

which is tautologically equivalent to
\( q \text{ is a power of two} \lor \forall x \leq q [2 \uparrow x \neq q] \)

Then 129<;1 is tautologically equivalent to
\( 1 \text{ is a power of two} \lor \forall x \leq 1 [2 \uparrow x \neq 1] \)

This is a variable-free formula and the ;0 in 129<;1;0 is an implicit use of T90. That is, 129<;1;0 is
\( 1 \text{ is a power of two} \lor [0 \leq 1 \rightarrow 2 \uparrow 0 \neq 1] \)

In general, \( \forall x \leq b A \) obeys the expected rule: we can substitute any term c, indicated by ;c, and obtain \( c \leq b \rightarrow A_x(c) \).
t131. \( 2 \uparrow x \) is a power of two

\[ \text{Proof. } \mathbf{H}_x 116 \quad 129^{<}; 2 \uparrow x; 128 \uparrow x \quad 5; 2 \uparrow x \]

t132. \( x \cdot y = 0 \Rightarrow x = 0 \lor y = 0 \)

\[ \text{Proof. } \mathbf{H}_x y 22; y \quad 99; x; P_y \quad 15; x; x; P_y \]

t133b. \( 2 \uparrow 0 \neq 0 \)

\[ \text{Proof. } \mathbf{H} 116 \quad 115 \quad 126; 2 \quad 3; 1 \]

t133i. \( 2 \uparrow x \neq 0 \Rightarrow 2 \uparrow Sx \neq 0 \)

\[ \text{Proof. } \mathbf{H}_x 116 \quad 115 \quad 126; 2; x \quad 132; 2; 2 \uparrow x \quad 3; 1 \]

t133. \( 2 \uparrow x \neq 0 \)

t134. \( \neg 0 \) is a power of two

\[ \text{Proof. } \mathbf{H} 129^{>}; 0; x \quad 133; x \]

Here is the explanation of the notation \( 129^{>}; 0; x \) in the proof. The formula \( 129^{>} \) is

\[ \neg q \text{ is a power of two } \lor \exists x \leq q [2 \uparrow x = q] \]

so \( 129^{>}; 0 \) is

\[ \neg 0 \text{ is a power of two } \lor \exists x \leq 0 [2 \uparrow x = 0] \]

Let \( x \) abbreviate \( \mu_{SS0 \uparrow x = 0}(0) \) and implicitly use T89. Then \( 129^{>}; 0; x \) is

\[ \neg 0 \text{ is a power of two } \lor [x \leq 0 \land 2 \uparrow x = 0] \]

In general, \( \exists x \leq b A \) obeys the expected rule: we can choose any \( y \) not previously used in the proof, hold it fixed, indicated by \( :y \), and obtain \( y \leq b \land A_x(y) \).

t135. \( q \) is a power of two \( \Rightarrow 2 \cdot q \) is a power of two

\[ \text{Proof. } \mathbf{H}_q 129^{>}; q; x \quad 129^{<}; 2; q; Sx \quad 126; 2; x \quad 128; Sx \]

t136b. \( x \uparrow (y + 0) = (x \uparrow y) \cdot (x \uparrow 0) \)

\[ \text{Proof. } \mathbf{H}_x y 12; y \quad 126; x \quad 117; x \uparrow y \quad 105; x \uparrow y; 1 \]

t136i. \( x \uparrow (y + z) = (x \uparrow y) \cdot (x \uparrow z) \Rightarrow x \uparrow (y + Sz) = (x \uparrow y) \cdot (x \uparrow Sz) \)

\[ \text{Proof. } \mathbf{H}_x y z 12; y; z \quad 126; x; y + z \quad 126; x; z \quad 102; x; x \uparrow y; x \uparrow z \quad 102; x \uparrow y; x; \uparrow z \quad 105; x; x \uparrow y \]

t136. \( x \uparrow (y + z) = (x \uparrow y) \cdot (x \uparrow z) \)
t137. $x \neq 0 \rightarrow x < 2 \cdot x$

Proof. $H; x 22; x 125; P; x 124; P; x; x$

t138. $0 \leq x$

Proof. $H; x 97; x 71; 0; x$

t139. $x \neq 0 \rightarrow 1 \leq x$

Proof. $H; x 116 115 22; x 138; P; x 112; 0; P; x$

t140. $x \neq 0 \rightarrow 2 \leq 2 \uparrow x$

Proof. $H; x 116 115 127; x 114; 2 \uparrow x 139; x 112; 1; x 73; 2; 8; x; 2 \uparrow x$

t141a. $x \cdot z = y \cdot z \& x < y \& z \neq 0 \rightarrow x = y$

Proof. $H; x; z; y 108; x; y 106; x; y; x; z 132; y; x; z 12; x; z 120; x; z; 0; (y; x) \cdot z$

t141. $x \cdot z = y \cdot z \& z \neq 0 \rightarrow x = y$

Proof. $H; x; z; y 141; x; z; y 141; a; y; z; x 77; x; y$

t142. $x \leq y \rightarrow z \cdot x \leq z \cdot y$

Proof. $H; x; y; z 107; x; y; z 105; z; x 105; z; y$

t143. $x \neq 0 \rightarrow x < x \cdot 2$

Proof. $H; x 137; x 105; x; 2$

t144. $x < y \& y \leq z \rightarrow x < z$

Proof. $H; x; y; z 56 \Rightarrow; x; y 73; x; y; z 56 \Rightarrow; x; z 80; x; y$

t145. $x < y \rightarrow 2 \uparrow x < 2 \uparrow y$

Proof. $H; x; y 108; x; y 136; 2; x; y; x 140; y; x 142; 2; 2 \uparrow (y; x); 2 \uparrow x 133; x 143; 2 \uparrow x 144; 2 \uparrow x; (2 \uparrow x); 2 \uparrow y$

t146. $x \leq y \rightarrow 2 \uparrow x \leq 2 \uparrow y$

Proof. $H; x; y 61; x; y 145; x; y 56 \Rightarrow; 2 \uparrow x; 2 \uparrow y 60; 2 \uparrow x$

t147. $2 \uparrow x < 2 \uparrow y \rightarrow x < y$

Proof. $H; x; y 77; x; y 56 \Rightarrow; 2 \uparrow x; 2 \uparrow y 145; y; x 80; 2 \uparrow x; 2 \uparrow y 56 \Rightarrow; 2 \uparrow y; 2 \uparrow x$

t148a. $2 \uparrow x = 2 \uparrow y \rightarrow \neg x < y$

Proof. $H; x; y 145; x; y 56 \Rightarrow; 2 \uparrow x; 2 \uparrow y$

t148. $2 \uparrow x = 2 \uparrow y \rightarrow x = y$

Proof. $H; x; y 148a; x; y 148a; y; x 77; x; y$

t149. $\neg [q \text{ is a power of two} \& q' \text{ is a power of two} \& q < q' \& q' < 2 \cdot q]$

Proof. $H; q; q' 135; q 129 \Rightarrow; q; x 129 \Rightarrow; q'; y 129 \Rightarrow; 2 \cdot q; x 126; 2; x 147; x; y 147; y; z 148; S; x; z 111; x; y$

d150. $p_{150}(q, x) \leftrightarrow q \text{ is a power of two} \& q \leq Sx \& Sx < 2 \cdot q$

t151a. $p_{150}(q, x) \& p_{150}(q', x) \rightarrow \neg q < q'$

Proof. $H; q; x; q' 150 \Rightarrow; q; x 150 \Rightarrow; q'; x 149; q; q' 122; q'; S; x; 2; q 56 \Rightarrow; q; q'$
t151. $p_{150}(q, x) \land p_{150}(q', x) \rightarrow q = q'$

Proof. $H: q;x:q' \quad 151a:q;x:q' \quad 151a:q';x;q' \quad 77:q;q'$

t152. $p_{150}(q, x) \land SSx < 2 \cdot q \rightarrow p_{150}(q, Sx)$

Proof. $H: q;x \quad 150 \rightarrow \neg q;x \quad 150 \leq ;q;Sx \quad 63;Sx \quad 53;q;Sx;SSx$

t153. $p_{150}(q, x) \land \neg SSx < 2 \cdot q \rightarrow p_{150}(2 \cdot q, Sx)$

Proof. $H: q;x \quad 150 \rightarrow ;q;x \quad 150 \leq ;2;q;Sx \quad 135;q \quad 124;Sx;2-q \quad 114;Sx;2-q \quad 56 \leq ;SSx;2-q \quad 60;SSx$

t154. $x \leq y \rightarrow x \leq Sy$

Proof. $H: x;y \quad 63;y \quad 73;x;y;Sy$

t155. $p_{150}(q, x) \rightarrow \exists q_1 \leq SSx[p_{150}(q_1, Sx)]$

Proof. $H: q;x^A \quad A;q \quad 154;q;x \quad 150 \rightarrow ;q;x \quad 150 \rightarrow ;2;q;Sx \quad 154;q;Sx$

The formula $H: q;x$ is tautologically equivalent to $p_{150}(q, x) \land \forall q_1 \leq SSx[\neg p_{150}(q_1, Sx)]$. The superscript $A$ labels the remnant $\forall q_1 \leq SSx[\neg p_{150}(q_1, Sx)]$ for later substitution of values for $q_1$.

t156b. $\exists q \leq 1[p_{150}(q, 0)]$

Proof. $H: 1 \quad 116 \quad 115 \quad 150 \leq ;1;0 \quad 130 \quad 60;1 \quad 3;0 \quad 137;1$

t156i. $\exists q \leq SSx[p_{150}(q, x)] \rightarrow \exists q_1 \leq SSx[p_{150}(q_1, Sx)]$

Proof. $H: q;x^A \quad 154;q;x_1 \quad A;q_1$

Note that $\exists q \leq SSx[p_{150}(q, x)]$ is $\mu_A(Sx)$ where $A$ is $p_{150}(q, x)$, and $\exists q_1 \leq SSx[p_{150}(q_1, Sx)]$ is $\mu_{A_1}(Sx)$ where $A_1$ is $A_q(q_1)$. By the remark after R82 they are the same formula, so induction applies.

t156. $\exists q \leq SSx[p_{150}(q, x)]$

d157. $Qx = q \leftrightarrow p_{150}(q, x)$

By 151 (UC) and 156 (EC) (and Proposition 7).

t158. $Qx$ is a power of two $\land Qx \leq Sx \land Sx < 2 \cdot Qx$ $^*$

Proof. $H: x \quad 157 \rightarrow ;x;Qx \quad 150 \rightarrow ;Qx;x \quad 5;Qx$

$Qx$ expresses $\lfloor \log_2(x + 1) \rfloor$, the largest power of two less than $x + 1$.

t159. $q$ is a power of two $\land q \leq Sx \land Sx < 2 \cdot q \rightarrow q = Qx$ $^*$

Proof. $H: q;x \quad 150 \leq ;q;x \quad 157 \leq ;x;q$

e160. $Rx = Sx - Qx$

t161. $Sx = Qx + Rx$

Proof. $H: x \quad 158;x \quad 160;x \quad 68;Qx;Sx$

t162. $x \leq y \rightarrow \exists w \leq y[x + w = y \land w \neq 0]$

Proof. $H: x;y;y-x \quad 108;x;y \quad 98;x;y-x \quad 71;y-x;x$
t163. \( x + y = z \) \& \( y \neq 0 \) \( \rightarrow \) \( x < z \)

\[ \text{Proof.}\ H:x:y:z\ 56 \leq x:z\ 71; x:y\ 12; x \ 120; x:y:0 \]

\[ \text{Proof.}\ H:x:y:z\ 162;x+y+x+z:w\ 72;x:y:w\ 120;x+y+w;z\ 163;y:w:z \]

\[ \text{Proof.}\ H:x\ 161;x\ 158;x\ 118;Qx\ 164;Qx;Rx;Qx \]

\[ t166.\ Sx = Qx + Rx \& \ Qx \text{ is a power of two} \& \ Rx < Qx \]

\[ \text{Proof.}\ H:x\ 161;x\ 158;x\ 165;x \]

\[ t167.\ x \leq y \rightarrow \exists w \leq y [x + w = y] \]

\[ \text{Proof.}\ H:x:y:y-x\ 68;x:y\ 98;x:y-x\ 71;y-x:x \]

\[ t168.\ x + y \leq x + z \rightarrow y \leq z \]

\[ \text{Proof.}\ H:x:y:z\ 167;x+y+x+z:w\ 72;x:y:w\ 120;x+y+w;z\ 71;y:w \]

\[ \text{Proof.}\ H:x:y:z\ 167;x+y+x+z:w\ 72;x:y:w\ 120;x+y+w;z\ 71;y:w \]

\[ \text{Proof.}\ H:x:y:z\ 79;y+z;x+z\ 98;y:z\ 98;x:z\ 168;z:y:x\ 80;x:y \]

\[ t171.\ Sx = q + r \& \ q \text{ is a power of two} \& \ r < q \rightarrow q = Qx \& \ r = Rx \]

\[ \text{Proof.}\ H:x:q:r\ 71; q;r\ 118; q\ 98; q:r\ 170; r:q; q\ 159; q;x\ 120; Qx;r;Rr; 161;x \]

\[ \#11.\ \text{Introduce concatenation of strings and prove a few of its properties.} \]

\[ e172.\ x \oplus y = P(Sx \cdot Qy + Ry) \]

Why is \( \oplus \) called concatenation? As an example, consider the string 101 (i.e., the number \( x \) such that \( Sx \) in binary is 1101) and the string 01 (i.e., the number \( y \) such that \( Sy \) in binary is 101). In binary, \( Qy \) is 100, \( Ry \) is 1, and \( Sx \cdot Qy + Ry \) is 110101, so \( x \oplus y \) is the string 10101 (i.e., the number whose successor in binary is 110101).

String arithmetic is analogous to number arithmetic, with one zero, \( \epsilon \), but with two successors: \( x \mapsto x \oplus 0 \) and \( x \mapsto x \oplus 1 \). Concatenation is the string analogue of addition. We have founded string arithmetic on number arithmetic, but we need to develop it to the point that it becomes independent of this foundation.

\[ t173.\ x \neq 0 \& \ y < z \rightarrow y < x \cdot z \]

\[ \text{Proof.}\ H:x:y:z\ 22;x\ 104;Pz:x\ 98;Pz:x:z\ 71;z;Pz:z\ 144;y:z:x:z \]

\[ t174.\ x < y \rightarrow z + x < z + y \]

\[ \text{Proof.}\ H:x:y:z\ 98;z:x\ 98;z:y\ 170;x:y:z \]

\[ t175.\ r < q \& \ r' < q' \rightarrow r \cdot q' + r' < q \cdot q' \]

\[ \text{Proof.}\ H:r:q:r':q'\ 162;r;q:w\ 162;r':q':w'\ 101;r;r':w'\ 106;r:w:r'+w'\ 173;w:r':q'\ 174;r':w'\cdot r'+r\cdot w' \]
t176. $R_x \cdot Q_y + R_y < Q_x \cdot Q_y$

Proof. $H : x : y \quad 166 ; x \quad 166 ; y \quad 175 ; R_x ; Q_x ; R_y ; Q_y$

\[ q_1 = 2 \quad \& \quad q_2 = 2 \quad \Rightarrow \quad q_1 \cdot q_2 = 2 \]

\[ Q_x \neq 0 \]

Proof. $H : x \quad 158 ; x \quad 134$

\[ S(x \oplus y) = Sx \cdot Q_y + R_y \]

Proof. $H : x : y \quad 172 ; x : y \quad 22 ; Sx \cdot Q_y + R_y \quad 15 ; Sx \cdot Q_y ; R_y \quad 3 ; x \quad 178 ; y \quad 132 ; Sx \cdot Q_y$

\[ Q(x \oplus y) = Qx \cdot Q_y \quad \& \quad R(x \oplus y) = Rx \cdot Q_y + R_y \]

Proof. $H : x : y \quad 179 ; x : y \quad 166 ; x \quad 166 ; y \quad 106 ; Qx ; R_x ; Q_y \quad 72 ; Qx \cdot Q_y ; R_x \cdot Q_y ; R_y \quad 177 ; Qx \cdot Q_y \quad 176 ; x ; y \quad 171 ; x \oplus y ; Q_x \cdot Q_y ; R_x \cdot Q_y + R_y$

\[ Q \left( x \oplus (y \oplus z) \right) = Q \left( (x \oplus y) \oplus z \right) \]

Proof. $H : x ; y ; z \quad 180 ; y ; z \quad 180 ; x ; y \quad 180 ; x \oplus y ; z \quad 180 ; x \oplus y ; z \quad 102 ; Qx ; Q_y ; R_y ; Q_y ; Q_y$

\[ R \left( x \oplus (y \oplus z) \right) = R \left( (x \oplus y) \oplus z \right) \]

Proof. $H : x ; y ; z \quad 180 ; y ; z \quad 180 ; x ; y \quad 180 ; x \oplus y ; z \quad 180 ; x \oplus y ; z \quad 102 ; R_x ; Q_y ; Q_y \quad 106 ; R_x \cdot Q_y \cdot R_y ; Q_y \quad 72 ; R_x \cdot Q_y \cdot Q_y \cdot R_y \cdot Q_y ; R_y \cdot Q_y ; R_z$

\[ x \oplus (y \oplus z) = (x \oplus y) \oplus z \quad \star \]

Proof. $H : x ; y ; z \quad 181 ; x ; y ; z \quad 182 ; x ; y ; z \quad 166 ; x \oplus (y \oplus z) \quad 166 ; (x \oplus y) \oplus z \quad 4 ; (x \oplus y) \oplus z \quad x \oplus (y \oplus z)$

\[ x_1 \leq y_1 \quad \& \quad x_2 \leq y_2 \quad \Rightarrow \quad x_1 + x_2 \leq y_1 + y_2 \]

Proof. $H : x_1 ; y_1 ; x_2 ; y_2 \quad 167 ; x_1 ; y_1 ; w_1 \quad 167 ; x_2 ; y_2 ; w_2 \quad 72 ; x_1 ; y_1 ; x_2 \oplus w_2 \quad 72 ; w_1 ; x_2 \oplus w_2 \quad 98 ; w_1 ; x_2 \quad 72 ; x_2 ; w_1 \quad 71 ; x_1 + x_2 \cdot w_1 + w_2$

With our identification of strings with numbers, 0 is the empty string, 1 is the zero bit, and 2 is the one bit. For greater readability, introduce new notation for these objects emphasizing their role as strings.

\[ \epsilon = 0 \quad \star \]

\[ 0 = 1 \quad \star \]

\[ 1 = 2 \quad \star \]

\[ 0 \neq \epsilon \quad \& \quad 1 \neq \epsilon \quad \& \quad 0 \neq 1 \quad \star \]

Proof. $H \quad 116 \quad 115 \quad 185 \quad 186 \quad 187 \quad 3 ; 0 \quad 3 ; 1 \quad 4 ; 0 \quad 1$

\[ Q \epsilon = 1 \quad \& \quad R \epsilon = 0 \]

Proof. $H \quad 116 \quad 115 \quad 185 \quad 12 ; 1 \quad 125 ; 0 \quad 130 \quad 171 ; 0 ; 1 ; 0$

\[ 2 \text{ is a power of two} \]

Proof. $H \quad 116 \quad 115 \quad 131 ; 1 \quad 126 ; 2 ; 0 \quad 117 ; 2 \quad 105 ; 2 ; 1$

\[ Q0 = 2 \quad \& \quad R0 = 0 \]

Proof. $H \quad 116 \quad 115 \quad 186 \quad 12 ; 2 \quad 190 \quad 125 ; 0 \quad 125 ; 1 \quad 56 \Rightarrow ; 0 ; 1 \quad 122 ; 0 ; 1 ; 2 \quad 171 ; 1 ; 2 ; 0$
t192. \( Q_1 = 2 \quad \& \quad R_1 = 1 \)

Proof. \( H \quad 116 \quad 115 \quad 187 \quad 12;2:0 \quad 190 \quad 125;1 \quad 171;2:2;1 \)

\[ Q_2 = Q_y \quad \& \quad R_2 = R_y \rightarrow x = y \]

Proof. \( H : x ; y \quad 166 ; x \quad 166 ; y \quad 4 ; x ; y \)

\[ \varepsilon \oplus x = x \]

Proof. \( H : x \quad 193 ; \varepsilon \oplus ; x ; x \quad 189 \quad 180 ; \varepsilon ; x \quad 117 ; Q_x \quad 103 ; Q_x \quad 97 ; R_x \)

\[ x \cdot 1 = x \]

Proof. \( H : x \quad 117 ; x \quad 105 ; x ; 1 \)

\[ x \oplus \varepsilon = x \]

Proof. \( H : x \quad 193 ; x \oplus ; \varepsilon ; x \quad 189 \quad 180 ; x ; \varepsilon \quad 195 ; Q_x \quad 195 ; R_x \quad 12 ; R_x \)

t197. \( y \oplus x = z \oplus x \rightarrow y = z \)

Proof. \( H : y ; x ; z \quad 180 ; y ; x \quad 180 ; x ; z \quad 178 ; x \quad 141 ; Q_y ; Q_x ; Q_z \quad 119 ; R_y ; Q_x ; R_z ; Q_x \quad 141 ; R_y ; Q_x ; R_z \quad 193 ; y ; z \)

\[ x \cdot y = x \cdot z \quad \& \quad x \neq 0 \rightarrow y = z \]

Proof. \( H : x ; y ; z \quad 105 ; x ; y \quad 105 ; x ; z \quad 141 ; y ; x ; z \)

\[ x \oplus y = x \oplus z \rightarrow y = z \]

Proof. \( H : x ; y ; z \quad 180 ; x ; y \quad 180 ; x ; z \quad 178 ; x \quad 198 ; Q_x ; Q_y ; Q_z \quad 120 ; R_x \cdot Q_y ; R_y ; R_z \quad 193 ; y ; z \)

\[ x \leq 1 \rightarrow x = 0 \quad \vee \quad x = 1 \]

Proof. \( H : x \quad 116 \quad 115 \quad 167 ; x ; 1 ; w \quad 22 ; w \quad 12 ; x ; P_w \quad 4 ; x ; P_w ; 0 \quad 15 ; x ; P_w \)

\[ x \neq 0 \rightarrow y \leq x \cdot y \]

Proof. \( H : x ; y \quad 22 ; x \quad 104 ; P_x ; y \quad 98 ; P_x ; y ; y \quad 71 ; y ; P_x ; y \)

\[ x \cdot y = 1 \rightarrow x = 1 \quad \& \quad y = 1 \]

Proof. \( H : x ; y \quad 116 \quad 115 \quad 103 ; y \quad 105 ; x ; y \quad 103 ; x \quad 201 ; x ; y \quad 201 ; y ; x \quad 200 ; x \quad 200 ; y \quad 3 ; 0 \)

\[ Q_2 = 1 \rightarrow x = \epsilon \]

Proof. \( H : x \quad 116 \quad 115 \quad 185 \quad 12 ; 1 \quad 166 ; x \quad 200 ; R_x \quad 56 \rightarrow ; R_x ; 1 \quad 4 ; x ; 0 \)

\[ x \oplus y = \epsilon \rightarrow x = \epsilon \quad \& \quad y = \epsilon \]

Proof. \( H : x ; y \quad 180 ; x ; y \quad 189 \quad 202 ; Q_x ; Q_y \quad 203 ; x \quad 203 ; y \)

r205. Parity 0 = 0 \quad \& \quad Parity Sx = C(Parity x, 1, 0)

Parity 0 expresses that \( x \) is even, and Parity \( x \) is S0 that \( x \) is odd.

t206. Parity \( x = 0 \rightarrow \) Parity \( Sx = 1 \)

Proof. \( H : x \quad 205 ; x \quad 33 ; 1 ; 0 ; 0 \)

T207. Parity \( x = 1 \rightarrow \) Parity \( Sx = 0 \)

Proof. \( H : x \quad 116 \quad 115 \quad 205 ; x \quad 33 ; 1 ; 0 ; 0 \)
t208. \( \text{Parity } 0 = 0 \) \& \( \text{Parity } 1 = 1 \) \& \( \text{Parity } 2 = 0 \) \& \( \text{Parity } \epsilon = 0 \) \& \( \text{Parity } \bot = 1 \) \& \( \text{Parity } \frac{1}{2} = 0 \)

Proof. H 116 115 205;0 205;1 185 186 187 33;1;0;0

\[ t209. \text{Parity } x = 0 \lor \text{Parity } x = 1 \]

Proof. H: \( x \) 116 115 205;0 \( P \) x 22;\( x \) 39;Parity \( P \)

\[ t210. \text{Parity } x = 0 \rightarrow \text{Parity} (x + 0) = \text{Parity } 0 \]

Proof. H: 208 12;\( x \)

\[ t210i. \ [\text{Parity } x = 0 \rightarrow \text{Parity} (x + y) = \text{Parity } y] \rightarrow [\text{Parity } x = 0 \rightarrow \text{Parity} (x + Sy) = \text{Parity } Sy] \]

Proof. H: \( x: y \) 12;\( x:y \) 205;\( x+y \) 205;\( y \)

\[ t210. \text{Parity } x = 0 \rightarrow \text{Parity} (x + y) = \text{Parity } y \]

\[ t211b. \text{Parity } x = 0 \rightarrow \text{Parity} (x \cdot 0) = 0 \]

Proof. H: \( x \) 100;\( x \) 205

\[ t211i. \ [\text{Parity } x = 0 \rightarrow \text{Parity} (x \cdot y) = 0] \rightarrow [\text{Parity } x = 0 \rightarrow \text{Parity} (x \cdot Sy) = 0] \]

Proof. H: \( x: y \) 100;\( x: \) 208 105;\( x:2 \) 211;\( x: \) 210;\( 2:y \)

\[ t211. \text{Parity } x = 0 \rightarrow \text{Parity} (x \cdot y) = 0 \]

\[ t212. \text{Parity} (x \cdot 2 + y) = \text{Parity } y \]

Proof. H: \( x: y \) 208 105;\( x:2 \) 211;\( 2:x \) 210;\( 2:y \)

\[ t213. \text{Parity} (x \cdot 2) = 0 \]

Proof. H: \( x \) 212;\( x:0 \) 208 12;\( x:2 \)

\[ t214. \text{Parity} (x \cdot 2 + 1) = 1 \]

Proof. H: \( x \) 212;\( x:1 \) 208

\[ t215. \text{Parity} R(x \oplus \bot) = 0 \]

Proof. H: \( x \) 180;\( x:0 \) 191 212;Rx:0 205

\[ t216. \text{Parity} R(x \oplus \frac{1}{2}) = 1 \]

Proof. H: \( x \) 180;\( x:1 \) 192 212;Rx:1 208

\[ t217. \ x \oplus \bot \neq y \oplus \frac{1}{2} \]

Proof. H: \( x: y \) 116 115 215;\( x \) 216;\( y \) 3:0

\[ \#12. \text{Construct the unary PR function symbol Chop that deletes the last bit, if any, of a string.} \]

r218. \( \text{Half } 0 = 0 \) \& \( \text{Half } Sx = C(\text{Parity } x, \text{Half } x, \text{S Half } x) \)

Half \( x \) expresses \( \lfloor x/2 \rfloor \). It deletes the rightmost binary bit of \( x \) (if \( x \) is not 0 or 1).

\[ t219. \text{Parity } x = 0 \rightarrow \text{Half } Sx = \text{Half } x \]

Proof. H: \( x \) 218;\( x \) 33;\( \text{Half } x: \text{S Half } x \)
t220. Parity $x = 1 \rightarrow \operatorname{Half} Sx = S \operatorname{Half} x$

Proof. \(H: x 116 \ 115 \ 218; x \ 33; \operatorname{Half} x; S \operatorname{Half} x: 0\)

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t221. Half $0 = 0 \& \operatorname{Half} 1 = 0 \& \operatorname{Half} 2 = 1 \& \operatorname{Half} S2 = 1$

Proof. \(H: x 116 \ 115 \ 218; 0 \ 218; 1 \ 218; 2 \ 208 \ 219; x \ 33; 0; 1; 0 \ 33; 0; 2; 1 \ 33; 1; 2; 0\)

---

d222. \(p_{222}(x) \leftrightarrow [\text{Parity } x = 0 \rightarrow x = 2 \cdot \operatorname{Half} x] \& [\text{Parity } x = 1 \rightarrow x = 2 \cdot \operatorname{Half} x + 1]\)

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t223b. \(p_{222}(0)\)

Proof. \(H: x 116 \ 115 \ 222; 0 \ 205 \ 221 \ 99; 2 \ 3; 0\)

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t223i. \(p_{222}(x) \rightarrow p_{222}(Sx)\)

Proof. \(H: x 116 \ 115 \ 222; x 222; 0; 3; 0 \ 12; 2 \cdot \operatorname{Half} x; 0 \ 206; x \ 207; x \ 219; x \ 220; x \ 12; 2; \operatorname{Half} x; 1 \ 100; 2; \operatorname{Half} x\)

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t223. \(p_{222}(x)\)

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t224. Parity $x = 0 \rightarrow x = 2 \cdot \operatorname{Half} x$

Proof. \(H: x 223; x \ 222; x\)

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t225. Parity $x = 1 \rightarrow x = 2 \cdot \operatorname{Half} x + 1$

Proof. \(H: x 223; x \ 222; x\)

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t226. $x \leq 2 \cdot \operatorname{Half} x + 1$

Proof. \(H: x 224; x \ 225; x \ 71; 2; \operatorname{Half} x; 1 \ 209; x \ 60; x\)

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t227. Half $x = 0 \rightarrow x = 0 \lor x = 1$

Proof. \(H: x 226; x \ 100; 2 \ 97; 1 \ 200; x\)

---

t228. \(x \neq \epsilon \rightarrow \operatorname{Half} Qx + \operatorname{Half} Rx \neq 0\)

Proof. \(H: x 15; \operatorname{Half} Qx; \operatorname{Half} Rx \ 227; Qx \ 178; x \ 203; x\)

---

e229. Chop $x = P(\operatorname{Half} Qx + \operatorname{Half} Rx)$

Chop $x$ deletes the rightmost bit, if any, of the string $x$. It is the string analogue of P.

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t230. \(\operatorname{Chop} \epsilon = \epsilon\)

Proof. \(H: \operatorname{Chop} \epsilon 189 \ 185 \ 221 \ 12; 0 \ 16\)

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t231. \(x \neq \epsilon \rightarrow \operatorname{SChop} x = \operatorname{Half} Qx + \operatorname{Half} Rx\)

Proof. \(H: \operatorname{Chop} \epsilon 229; x \ 228; x \ 22; \operatorname{Half} Qx + \operatorname{Half} Rx\)

---

t232. \(2 \cdot \operatorname{Half} x \leq x\)

Proof. \(H: x 224; x \ 225; x \ 60; x \ 71; 2; \operatorname{Half} x; 1 \ 209; x\)

---

t233. Parity $q = 0 \& r < q \rightarrow \operatorname{Half} r < \operatorname{Half} q$

Proof. \(H: q; r 79; \operatorname{Half} q; \operatorname{Half} r 224; q 118; \operatorname{Half} q; 184; \operatorname{Half} q; \operatorname{Half} r; \operatorname{Half} q; \operatorname{Half} r; 232; x 118; \operatorname{Half} r; 73; q; \operatorname{Half} r; \operatorname{Half} r; \operatorname{Half} r; r 80; r; q\)
t234. Parity($2 \uparrow Sx$) = 0
   \textbf{Proof.} H: \quad x \quad 126;2; x \quad 208 \quad 211;2; \uparrow x

t235. Half($2 \uparrow Sx$) = 2 \uparrow x
   \textbf{Proof.} H: \quad x \quad 116 \quad 115 \quad 234; x \quad 224;2; \uparrow Sx \quad 126;2; x \quad 3;1 \quad 198;2; \text{Half}(2 \uparrow Sx); 2 \uparrow x

t236. $q$ is a power of two & $q \neq 1$ \rightarrow Parity $q$ = 0
   \textbf{Proof.} H: \quad q \quad 129 \rightarrow q(x); x \quad 126;2; x \quad 22; x \quad 134 \quad 234; P x

t237. $x \neq 0$ \rightarrow Half($2 \uparrow x$) is a power of two
   \textbf{Proof.} H: \quad x \quad 116 \quad 115 \quad 22; x \quad 235; P x \quad 131; P x

t238. $x \neq \epsilon$ \rightarrow Half $Qx$ is a power of two
   \textbf{Proof.} H: \quad x \quad 158; x \quad 116 \quad 115 \quad 129 \rightarrow q(x); y \quad 126; 2 \quad 203; x \quad 237; y

t239. $x \neq \epsilon$ \rightarrow Q Chop $x$ = Half $Qx$ & R Chop $x$ = Half $Rx$
   \textbf{Proof.} H: \quad x \quad 171; Chop $x; Half Qx; Half Rx; 231; x \quad 238; x \quad 236; Qx \quad 166; x \quad 203; x \quad 233; Q x; Rx$

t240. Chop $0$ = $\epsilon$ & Chop $1$ = $\epsilon$
   \textbf{Proof.} H: \quad 242; x \quad 180; x \quad 191 \quad 192 \quad 221 \quad 203; Chop $0$ \quad 203; Chop $1$

t241b. Half($0 \cdot 2$) = 0 & Half($0 \cdot 2 + 1$) = 0
   \textbf{Proof.} H: \quad 103; 2 \quad 97; 1 \quad 221

t241i. Half($x \cdot 2 + 1$) = $x$ & Half($x \cdot 2$) = $x$ \rightarrow Half($Sx \cdot 2 + 1$) = $Sx$ & Half($Sx \cdot 2$) = $Sx$
   \textbf{Proof.} H: \quad x \quad 116 \quad 115 \quad 104; x; 2 \quad 12; x; 2; 1 \quad 212; x; 1 \quad 208 \quad 220; x; 2 + 1 \quad 12; x; 2 + 2; 0 \quad 12; S(x; 2 + 2) \quad 212; x; 2 \quad 219; x; 2 + 2

t241. Half($x \cdot 2 + 1$) = $x$ & Half($x \cdot 2$) = $x$

t242. $x \oplus 0 \neq \epsilon$ & $x \oplus 1 \neq \epsilon$
   \textbf{Proof.} H: \quad x \quad 204; x; 0 \quad 204; x; 1 \quad 188

t243. Chop($x \oplus 0$) = $x$
   \textbf{Proof.} H: \quad 242; x \quad 239; x; \oplus 0 \quad 180; x; 0 \quad 191 \quad 241; Qx \quad 241; Rx \quad 12; Rx \cdot 2 \quad 193; x; Chop($x \oplus 0$)

t244. Chop($x \oplus 1$) = $x$
   \textbf{Proof.} H: \quad 242; x \quad 239; x; \oplus 1 \quad 180; x; 1 \quad 192 \quad 241; Qx \quad 241; Rx \quad 12; Rx \cdot 2 \quad 193; x; Chop($x \oplus 1$)

t245. $x \neq \epsilon$ \rightarrow Q(Chop $x \oplus 0$) = Qx & Q(Chop $x \oplus 1$) = Qx
   \textbf{Proof.} H: \quad 180; Chop $x; \oplus 0 \quad 180; Chop x; \oplus 1 \quad 191 \quad 192 \quad 158; x \quad 203; x \quad 236; Qx \quad 224; Qx \quad 239; x \quad 105; 2; Q \text{Chop} x$

t246. Parity $x$ = Parity $y$ & Half $x$ = Half $y$ \rightarrow $x$ = $y$
   \textbf{Proof.} H: \quad x; y \quad 224; x \quad 224; y \quad 225; x \quad 225; y \quad 221; x \quad 209; x \quad ? \text{Parity} x = 0

t247. $x \neq \epsilon$ \rightarrow R(Chop $x \oplus 0$) = Half $Rx \cdot 2$
   \textbf{Proof.} H: \quad 180; Chop $x; \oplus 0 \quad 191 \quad 12; R \text{Chop} x \cdot 2 \quad 239; x$

t248. $x \neq \epsilon$ & Parity $Rx = 0$ \rightarrow R(Chop $x \oplus 0$) = $Rx$
   \textbf{Proof.} H: \quad x \quad 247; x \quad 241; Half Rx \quad 213; Half $Rx \quad 246; R(\text{Chop} x; \oplus 0); R x$
t249. \( x \neq \epsilon \rightarrow R(\text{Chop} x \oplus 1) = \text{Half} R x \cdot 2 + 1 \)

**Proof.** \( H: x \quad 180; \text{Chop} x \downarrow 192 \quad 239; x \)

\( x \neq \epsilon \) \& \( \text{Parity} R x = 1 \rightarrow R(\text{Chop} x \oplus 1) = R x \)

**Proof.** \( H: x \quad 249; x \quad 241; \text{Half} R x \quad 214; \text{Half} R x \quad 246; R(\text{Chop} x \oplus 1); R x \)

\( x \neq \epsilon \) \& \( \text{Parity} R x = 0 \rightarrow x = \text{Chop} x \oplus 0 \)

**Proof.** \( H: x \quad 245; x \quad 248; x \quad 193; x; \text{Chop} x \downarrow 0 \)

\( x \neq \epsilon \) \& \( \text{Parity} R x = 1 \rightarrow x = \text{Chop} x \oplus 1 \)

**Proof.** \( H: x \quad 245; x \quad 250; x \quad 193; x; \text{Chop} x \downarrow 1 \)

\( x \neq \epsilon \rightarrow x = \text{Chop} x \oplus 0 \lor x = \text{Chop} x \oplus 1 \)

**Proof.** \( H: x \quad 251; x \quad 252; x \quad 209; R x \)

\( y \neq \epsilon \rightarrow \text{Chop}(x \oplus y) = x \oplus \text{Chop} y \)

**Proof.** \( H: y; x \quad 253; y \quad 183; x; \text{Chop} y \downarrow 0 \quad 183; x; \text{Chop} y \downarrow 1 \quad 243; x; \text{Chop} y \quad 244; x; \text{Chop} y \)

#13. Establish string recursion and string induction.

**Proposition 8.** Let \( \vec{x}, y, \) and \( z \) be distinct, let \( a \) contain no variables other than those in \( \vec{x} \), and let \( b \) and \( c \) contain no variables other than those in \( \vec{x}, y, \) and \( z \). Define \( f \) by primitive recursion by cases:

\[
(6) \quad f(\vec{x}, 0) = a \quad \& \quad f(\vec{x}, Sy) = \\
C(\chi [Sy = \text{Chop} Sy \oplus 0], b_z(\text{Chop} Sy), C(\chi [Sy = \text{Chop} Sy \oplus 1], c_z(\text{Chop} Sy), 0))
\]

Then

\[
(7) \quad f(\vec{x}, \epsilon) = a \quad \& \quad f(\vec{x}, w \oplus 0) = b_z(w) \quad \& \quad f(\vec{x}, w \oplus 1) = c_z(w)
\]

**Proof.** We have \( f(\vec{x}, \epsilon) = a \) by 185. By Proposition 4 of §4,

\[
(8) \quad \text{Sy} = \text{Chop} Sy \oplus 0 \rightarrow f(\vec{x}, \text{Sy}) = \text{b}_z(\text{Chop} \text{Sy})
\]

\[
(9) \quad \neg [\text{Sy} = \text{Chop} Sy \oplus 0] \quad \& \quad [\text{Sy} = \text{Chop} Sy \oplus 1] \rightarrow f(\vec{x}, \text{Sy}) = \text{c}_z(\text{Chop} \text{Sy})
\]

\[
(10) \quad \neg [\text{Sy} = \text{Chop} Sy \oplus 0] \quad \& \quad \neg [\text{Sy} = \text{Chop} Sy \oplus 1] \rightarrow f(\vec{x}, \text{Sy}) = 0
\]

By 217; \text{Chop} \text{Sy}; \text{Chop} \text{Sy} and (9),

\[
(11) \quad \text{Sy} = \text{Chop} Sy \oplus 1 \rightarrow f(\vec{x}, \text{Sy}) = \text{c}_z(\text{Chop} \text{Sy})
\]

Note that (10), the “else” clause, is irrelevant, since its hypothesis cannot hold, by 253; \text{Sy} together with 3; \text{y} and 185. Now consider (8); \text{P}(w \oplus 0):

\[
\text{SP}(w \oplus 0) = \text{Chop} \text{SP}(w \oplus 0) \oplus 0 \rightarrow f(\vec{x}, \text{SP}(w \oplus 0)) = \text{b}_z(\text{Chop} \text{SP}(w \oplus 0))
\]
We have \( w \oplus 0 \neq 0 \) (by 185 and 204; \( w;0 \) and 188), so \( \text{SP}(w \oplus 0) = w \oplus 0 \) by 22; \( w \oplus 0 \). Also, \( \text{Chop}(w \oplus 0) = w \) by 243; \( w \). Therefore

\[
w \oplus 0 = w \oplus 0 \rightarrow f(\vec{x}, w \oplus 0) = b_z(w)
\]

and so \( f(\vec{x}, w \oplus 0) = b_z(w) \). The derivation of \( f(\vec{x}, w \oplus 1) = c_z(w) \) from (11); \( \text{SP}(w \oplus 1) \) is entirely similar. Hence (7). □

A string recursion is a primitive recursion of the form (6), but string recursions will be introduced simply by (7).

\[t255. \quad x < x \oplus 0\]

**Proof.** \( H: x \quad 172; x; 0 \quad 191 \quad 12; Sx; 2 \quad 105; Sx; 2 \quad 118; Sx \quad 12; Sx; x \quad 16; Sx + x \quad 71; Sx; x \quad 125; x \quad 144; x; Sx; Sx + x\)

\[t256. \quad x + 1 = Sx\]

**Proof.** \( H: x \quad 115 \quad 12; x; 0\)

\[t257. \quad x < x \oplus 1\]

**Proof.** \( H: x \quad 172; x; 1 \quad 192 \quad 256; Sx; 2 \quad 16; Sx; 2 \quad 105; Sx; 2 \quad 118; Sx \quad 125; x \quad 144; x; Sx; Sx + Sx\)

\[t258. \quad x \neq \epsilon \rightarrow \text{Chop} \ x < x\]

**Proof.** \( H: x \quad 253; x \quad 255; \text{Chop} \ x \quad 257; \text{Chop} \ x\)

**Proposition 9.** If \( \vdash A_x(\epsilon) \) and \( \vdash A_x(x') \rightarrow A_x(x' \oplus 0) \) and \( \vdash A_x(x') \rightarrow A_x(x' \oplus 1) \), then \( \vdash A_x(x) \).

**Proof.** Suppose that

\[
(12) \quad A_x(\epsilon)
\]

\[
(13) \quad A_x(x') \rightarrow A_x(x' \oplus 0)
\]

\[
(14) \quad A_x(x') \rightarrow A_x(x' \oplus 1)
\]

and use the least number principle (Proposition 6 of §5). Suppose that there is a least counterexample \( z \) to \( A \):

\[
(15) \quad \neg A_x(z) \quad \& \quad \forall y < z[A_x(y)]
\]

We have \( z \neq \epsilon \) by (12). Then, by 253; \( z \),

\[
(16) \quad z = \text{Chop} \ z \oplus 0 \quad \vee \quad z = \text{Chop} \ z \oplus 1
\]

By (15); \( z \) ; \( \text{Chop} \ z \) and 258; \( z \),

\[
A_x(\text{Chop} \ z)
\]

The first alternative of (16) does not hold, by (13); \( \text{Chop} \ z \), and the second does not hold, by (14); \( \text{Chop} \ z \). Hence \( A_x(x) \) by the least number principle. □

The derived rule of inference, from \( A_x(\epsilon) \) and \( A_x(x') \rightarrow A_x(x' \oplus 0) \) and \( A_x(x') \rightarrow A_x(x' \oplus 1) \) infer \( A_x(x) \), is string induction.

**#14.** Construct Length by string recursion, and introduce \( \preceq \) (shorter than) and \( \prec \) (strictly shorter than). Prove \( x \preceq y \rightarrow x \leq 2 \cdot Qx \) and \( x \prec y \rightarrow x < y \). The first will enable bounded quantifiers with bounding symbol \( \preceq \) rather than \( \leq \), and the second will enable the use of the least number principle for strings (“the shortest string principle”). Express the \( i \)’th bit of a string.
Sometimes, as here, the two string induction steps are combined into a single formula for the bounding symbol. Length(x ⊕ 0) = Length(x) & Length(x ⊕ 1) = Length(x) ⊕ y

\textbf{Proof.} H: x 196;x 259 12;Length x

Length(x ⊕ y) = Length x + Length y  \rightarrow  Length(x ⊕ (y ⊕ 0)) = Length x + Length(y ⊕ 0) & Length(x ⊕ (y ⊕ 1)) = Length x + Length(y ⊕ 1)

\textbf{Proof.} H:x:y 259;y 183;x:y:0 183;x:y:1 259;x:2 12;Length x;Length y

Length(x ⊕ y) = Length x + Length y

When tξ immediately follows tξb, tξi, and tξj, it is an inference by string induction. Sometimes, as here, the two string induction steps are combined into a single formula tξij.

Qx = 2 \uparrow (Length x)

\textbf{Proof.} H 259 126;2 189

Qx = 2 \uparrow Length x  \rightarrow  Q(x \oplus 0) = 2 \uparrow Length(x \oplus 0) & Q(x \oplus 1) = 2 \uparrow Length(x \oplus 1)

\textbf{Proof.} H:x 259;x 126;2;Length x 180;x:0 180;x:1 191 192 105;Q;x:2

Qx = 2 \uparrow Length x

x ≤ Sy  \rightarrow  x ≤ y  \lor  x = Sy

\textbf{Proof.} H:x:y 68;x:Sy 22;Sy−x 12;x:P(Sy−x) 4;y+x+P(Sy−x) 71;x:P(Sy−x)

x ≤ 0  \rightarrow  2 \uparrow x ≤ 2 \uparrow 0

\textbf{Proof.} H:x 57;x 60;2\uparrow 0

[x ≤ y  \rightarrow  2 \uparrow x ≤ 2 \uparrow y]  \rightarrow  [x ≤ Sy  \rightarrow  2 \uparrow x ≤ 2 \uparrow Sy]

\textbf{Proof.} H:x:y 262;x:y 60;2\uparrow x 126;2;y 118;2\uparrow y 71;2\uparrow y;2\uparrow y 73;2\uparrow x;2\uparrow y;2\uparrow 2\uparrow y

t63. x ≤ y  \rightarrow  2 \uparrow x ≤ 2 \uparrow y

d64. x ≤ y  \leftrightarrow  \text{Length} x ≤ \text{Length} y

d65. x < y  \leftrightarrow  \text{Length} x < \text{Length} y

t66. x ≤ y  \rightarrow  2 \cdot x ≤ 2 \cdot y

\textbf{Proof.} H:x:y 118;x 118;y 184;x:y;x:y

t67. x ≤ y  \rightarrow  x ≤ 2 \cdot Qy

\textbf{Proof.} H:x:y 264 \rightarrow;x:y 263;Length x;Length y 261;x 261;y 158;x 63;x 56 \rightarrow;Sx:2;Qx 73;x:Sy:2;Qx 266;Qx:Qy 71;x:2;Qx:2;Qy

Let s be a binary predicate symbol, written in infix notation. If for some term d containing no variable other than y we have ⊢ x s b A then we call s a \textit{bounding symbol} with \textit{bound} d. Thus x ≤ is a bounding symbol with bound 2 \cdot Qy by t67. For a bounding symbol s with bound d let ∃s b A abbreviate ∃x≤d y(b)[x s b & A] and ∀x s b A abbreviate ∀x≤d y(b)[x s b → A]. (The bound d is not unique, but we choose one, and it is implicit in the notations ∃s b A and ∀x s b A.) We have already used these abbreviations for the bounding symbol < (with bound y).
t268. \( \text{SR}x < Qx \rightarrow QSx = Qx \)

**Proof.** \( H:x \) 166; \( x \) 12; \( Qx:Rx \) 171; \( Sx:Qx:SRx \)

\( t269. \) \( q \) is a power of two \( \rightarrow 0 < q \)

**Proof.** \( H:q \) 134 58; \( q \) 56 \( \equiv \) 0; \( q \)

\( t270. \) \( \text{SR}x = Qx \rightarrow QSx = 2 \cdot Qx \)

**Proof.** \( H:x \) 166; \( x \) 190 177; \( 2:Qx \) 12; \( Qx:Rx \) 118; \( Qx \) 269; \( 2:Qx \) 12; \( 2:Qx \) 171; \( Sx:2:Qx:0 \)

\( t271. \) \( Qx \leq QSx \)

**Proof.** \( H:x \) 60; \( Qx \) 268; \( x \) 166; \( x \) 270; \( x \) 118; \( Qx \) 71; \( Qx:Qx \) 114; \( Rx:Qx \) 56 \( \equiv \) SRx; \( Qx \)

\( t272b. \) \( x \leq 0 \rightarrow Qx \leq Q0 \)

**Proof.** \( H:x \) 57; \( x \) 60; \( Q0 \)

\( t272i. \) \( [x \leq y \rightarrow Qx \leq Qy] \rightarrow [x \leq Sy \rightarrow Qx \leq QSy] \)

**Proof.** \( H:x:y \) 110; \( x:y \) 60; \( QSy \) 271; \( y \) 73; \( Qx:Qy:QSy \)

\( t272. \) \( x \leq y \rightarrow Qx \leq Qy \)

\( t273. \) \( x < y \rightarrow x < y \)

\(*\)

Since \( \text{Length} \ x < \text{Length} \ y \), we have \( Qx < Qy \) by \( t261 \) and \( t145 \). If \( x < y \), then \( y \leq x \), so \( Qy \leq Qx \) by \( t272 \), a contradiction. It is essential to have the strict bound \( y \) itself on \( \prec \) to apply the least number principle to strings.

**Proof.** \( H:x:y \) 265 \( \rightarrow \) \( x:y \) 145; \( \text{Length} \ x; \text{Length} \ y \) 261; \( x \) 261; \( y \) 272; \( y:x \) 80; \( Qx:Qy \) 79; \( y:x \)

\( t274. \) \( \epsilon \leq x \quad \& \quad x \leq x \)

**Proof.** \( H:x \) 264 \( \leq \) \( x \) \( x \) \( 264 \leq \) \( x \) \( x \) \( 259 \) 58; \( \text{Length} \ x \) 60; \( \text{Length} \ x \)

\( t275. \) \( x \leq y \quad \& \quad y \leq z \rightarrow x \leq z \)

**Proof.** \( H:x:y:z \) 264 \( \rightarrow \) \( x:y \) \( 264 \rightarrow \) \( y:z \) \( 73; \text{Length} \ x; \text{Length} \ y; \text{Length} \ z \) \( 264 \leq \) \( x:z \)

\( t276. \) \( x \oplus y \leq y \oplus x \)

**Proof.** \( H:x:y \) 260; \( x:y \) \( 260; \) \( y:x \) \( 98; \text{Length} \ x; \text{Length} \ y \) \( 60; \text{Length} \ x + \text{Length} \ y \) \( 264 \leq \) \( x:y:y:x \)

\( t277. \) \( x \leq x \oplus y \quad \& \quad y \leq x \oplus y \)

**Proof.** \( H:x:y \) 264 \( \leq \) \( x:y \) \( 264 \leq \) \( y:x \) \( y \) \( 71; \text{Length} \ x; \text{Length} \ y \) \( 98; \text{Length} \ x; \text{Length} \ y \) \( 71; \text{Length} \ y; \text{Length} \ x \)

\( t278. \) \( \text{Length} \ 0 = 1 \quad \& \quad \text{Length} \ 1 = 1 \)

**Proof.** \( H \) 259; \( \epsilon \) 194; \( 0 \) 194; \( 1 \) 115

\( t000b. \) \( x_1 \oplus \epsilon = x_2 \oplus \epsilon \rightarrow x_1 = x_2 \) same \( t000b \). \( H:x_1:x_2 \) 196; \( x_1 \) 196; \( x_2 \)

\( t000i. \) \( [x_1 \oplus y = x_2 \oplus y \rightarrow x_1 = x_2] \rightarrow [x_1 \oplus y \oplus 0 = x_2 \oplus y \oplus 0 \rightarrow x_1 = x_2] \)

**Proof.** \( H:x_1:y:x_2 \) 183; \( x_1:y:0 \) 183; \( x_2:y:0 \) 243; \( x_1:y \) 243; \( x_2:y \)

\( t000j. \) \( [x_1 \oplus y = x_2 \oplus y \rightarrow x_1 = x_2] \rightarrow [x_1 \oplus y \oplus 1 = x_2 \oplus y \oplus 1 \rightarrow x_1 = x_2] \)

**Proof.** \( H:x_1:y:x_2 \) 183; \( x_1:y:1 \) 183; \( x_2:y:1 \) 244; \( x_1:y \) 244; \( x_2:y \)
\[ x_1 \oplus y = x_2 \oplus y \quad \rightarrow \quad x_1 = x_2 \]
\[ \text{Length } x = 0 \quad \rightarrow \quad x = \varepsilon \]
\[ \text{Proof. } H: x \quad 253; \text{Chop } x; 0 \quad 260; \text{Chop } x; 1 \quad 278 \quad 15; \text{Length Chop } x; 1 \quad 115 \quad 3:0 \]
\[ d_{288}. \quad x \text{ ends with c } \quad \leftrightarrow \quad \exists a \leq x [x = a \oplus c] \]
\[ a \oplus c \text{ ends with c} \]
\[ \text{Proof. } H: a; c \quad 277; a; c \quad 288 \leq \vdash a \oplus c; a \quad 5 \vdash a \oplus c \]
\[ d_{290}. \quad x \text{ begins with a } \quad \leftrightarrow \quad \exists c \leq x [x = a \oplus c] \]
\[ a \oplus c \text{ begins with a} \]
\[ \text{Proof. } H: a; c \quad 277; a; c \quad 290 \vdash \vdash a \oplus c; a \quad 5 \vdash a \oplus c \]
\[ t_{292}. \quad x \text{ begins with a } \quad \rightarrow \quad x \oplus y \text{ begins with a} \]
\[ \text{Proof. } H: x; a; y \quad 290 \vdash \vdash x; a; c \quad 183; a; c; y \quad 291; a; c \oplus y \]
\[ t_{293}. \quad y \text{ ends with c } \quad \rightarrow \quad x \oplus y \text{ ends with c} \]
\[ \text{Proof. } H: y; c; x \quad 288 \vdash \vdash y; c; a \quad 183; x; a; c \quad 289; x; a; c \]
\[ t_{294}. \quad x \text{ ends with } x \quad \& \quad x \text{ ends with } \varepsilon \quad \& \quad x \text{ begins with } x \quad \& \quad x \text{ begins with } \varepsilon \]
\[ \text{Proof. } H: x \quad 194; x \quad 196; x \quad 289; \varepsilon; x \quad 289; \varepsilon; x \quad 291; \varepsilon; x \quad 291; x; \varepsilon \]
\[ t_{002}. \quad \varepsilon \text{ begins with } x \quad \lor \quad \varepsilon \text{ ends with } x \quad \rightarrow \quad x = \varepsilon \]
\[ \text{Proof. } H: x \quad 290 \vdash \vdash x; c; c \quad 288 \vdash \vdash x; a; c \quad 204; x; a; x \]
\[ t_{005}. \quad x_1 + y_1 = x_2 + y_2 \quad \& \quad x_1 \leq x_2 \quad \rightarrow \quad y_2 \leq y_1 \]
\[ \text{Proof. } H: x_1; y_1; x_2; y_2 \quad 68; x_1; x_2 \quad 72; x_1; x_2 - x_1; y_2 \quad 120; x_1; y_1; (x_2 - x_1) + y_2 \quad 98; x_2 - x_1; y_2 \quad 71; y_2; x_2 - x_1 \]
\[ t_{006}. \quad a_1 \oplus c_1 = a_2 \oplus c_2 \quad \& \quad a_1 \leq a_2 \quad \rightarrow \quad c_2 \leq c_1 \]
\[ \text{Proof. } H: a_1; c_1; a_2; c_2 \quad 264 \vdash \vdash a_1; a_2 \quad 264 \vdash \vdash c_2; c_1 \quad 260; a_1; c_1 \quad 260; a_2; c_2 \quad 005; \text{Length } a_1; \text{Length } c_1; \text{Length } a_2; \text{Length } c_2 \]
\[ t_{008}. \quad x \leq \varepsilon \quad \rightarrow \quad x = \varepsilon \]
\[ \text{Proof. } H: x \quad 264 \vdash \vdash x; \varepsilon \quad 57; \text{Length } x \quad 001; x \quad 259 \]
\[ t_{007}. \quad a_1 \oplus c_1 = a_2 \oplus c_2 \quad \& \quad a_1 \leq a_2 \quad \& \quad c_2 \neq \varepsilon \quad \rightarrow \quad c_1 \neq \varepsilon \]
\[ \text{Proof. } H: a_1; c_1; a_2; c_2 \quad 006; a_1; c_1; a_2; c_2 \quad 008; c_2 \]
\[ t_{003b}. \quad \varepsilon \text{ begins with } a_1 \quad \& \quad \varepsilon \text{ begins with } a_2 \quad \& \quad a_1 \leq a_2 \quad \rightarrow \quad a_2 \text{ begins with } a_1 \]
\[ \text{Proof. } H: a_1; a_2 \quad 002; a_1 \quad 002; a_2 \quad 294; \varepsilon \]
\[ t_{003i}. \quad [x \text{ begins with } a_1 \quad \& \quad x \text{ begins with } a_2 \quad \& \quad a_1 \leq a_2 \quad \rightarrow \quad a_2 \text{ begins with } a_1] \]
\[ \rightarrow \quad [x \oplus 0 \text{ begins with } a_1 \quad \& \quad x \oplus 0 \text{ begins with } a_2 \quad \& \quad a_1 \leq a_2 \quad \rightarrow \quad a_2 \text{ begins with } a_1] \]
\[ \text{Proof. } \text{Suppose not. Then } \]
\[ 1 \quad x \text{ begins with } a_1 \& x \text{ begins with } a_2 \& a_1 \leq a_2 \rightarrow a_2 \text{ begins with } a_1 \]
\[ 2 \quad x \oplus 0 \text{ begins with } a_1 \]
\[ 3 \quad x \oplus 0 \text{ begins with } a_2 \]
\[ 4 \quad a_1 \leq a_2 \]
\[ t003j. \] \( x \) begins with \( a_1 \) \& \( x \) begins with \( a_2 \) \& \( a_1 \preceq a_2 \rightarrow a_2 \) begins with \( a_1 \)

\[ \rightarrow [x \oplus 1 \text{ begins with } a_1 \& x \oplus 1 \text{ begins with } a_2 \& a_1 \preceq a_2 \rightarrow a_2 \text{ begins with } a_1] \]

The proof is entirely similar: replace each \( 0 \) by \( 1 \) and 243 by 244.

t003. \( x \) begins with \( a_1 \) \& \( x \) begins with \( a_2 \) \& \( a_1 \preceq a_2 \rightarrow a_2 \) begins with \( a_1 \)

t010. \( x \) begins with \( y \) \& \( y \) begins with \( x \rightarrow x = y \)

Proof. \( H : x : y \)

\[ 290^\rightarrow : x : y : c \quad 290^\rightarrow : y : x : d \quad 183 : x : d : c \quad 196 : x \quad 199 : x : e : d \oplus c \quad 204 : d : e \]

t011. \( x \) ends with \( y \) \& \( y \) ends with \( x \rightarrow x = y \)

Proof. \( H : x : y \)

\[ 288^\rightarrow : x : y : c \quad 288^\rightarrow : y : x : d \quad 194 : x \quad 183 : d : x \quad 197 : c : x : e \oplus d \quad 204 : c : d \]

d012. last-bit \( (x, b) \leftrightarrow [x = \text{Chop } x \oplus 0 \& b = 0] \lor [x = \text{Chop } x \oplus 1 \& b = 1] \lor [x = \epsilon \& b = \epsilon] \)

t013. last-bit \( (x, b_1) \) \& last-bit \( (x, b_2) \rightarrow b_1 = b_2 \)

Proof. \( H : b_1 : b_2 \)

\[ 012^\rightarrow : x : b_1 \quad 012^\rightarrow : x : b_2 \quad 253 : x \quad x \neq \epsilon \]
Afterword on two works by Ed Nelson

Sam Buss and Terence Tao

Two of Ed Nelson’s unfinished papers, *Elements* (dated March 12, 2013) and *Inconsistency of Primitive Recursive Arithmetic* (undated, also known as the *Balrog* paper) have the goal of proving the inconsistency of number theory. As Nelson writes in *Elements*, “The aim of this work is to show that contemporary mathematics, including Peano arithmetic, is inconsistent ...”.

Neither of these papers have been circulated before in their current forms. An earlier version of *Elements* was circulated in 2011, but was found to have problems in its treatment of proofs generated by Chaitin machines. The new 2013 version uses a similar approach, but gives a much more detailed explanation of the planned proof, and it handles Chaitin machines differently so as to address the earlier problems.

Nelson’s remarkable program to establish the inconsistency of Peano arithmetic was intertwined with his development of Internal Set Theory [4] and especially Predicative Arithmetic [5]. Predicative Arithmetic is a constructive fragment of arithmetic, and Nelson’s development of Predicative Arithmetic was inspired in part by Yessenin-Volpin’s ultra-intuitionistic set theory [6]. The mathematical content of Predicative Arithmetic is closely tied to theories of bounded arithmetic such as IΔ₀, IΔ₀ + Ω₁, S₂¹ and T₂¹. Indeed, Nelson [5] independently discovered some of the important tools for bounded arithmetic, including the technique of speeding up induction on cuts and the local interpretability of predicative arithmetic and bounded arithmetic in Robinson’s theory Q. Nelson’s Predicative Arithmetic was also influential for the definition by one of us (Buss) of the theories S₂¹ and T₂¹ of bounded arithmetic, including notably the use of Nelson’s smash function.

The *Elements* manuscript gives a detailed, high-level outline of Nelson’s plan for a proof of the inconsistency of Peano arithmetic (and primitive recursive arithmetic). One of the principal tools is a novel use of a recent proof by Kritchman and Raz [3] of Godel’s second incompleteness theorem based on the “surprise examination”. Nelson also uses Kolmogorov complexity and techniques from cut-elimination. The detailed plan of the inconsistency proof is outlined as Steps 1-17 in section 9 near the end of *Elements*. The plan first discusses a system S* which is a predicative theory including bounded induction and which is strong enough to express concepts about metamathetical concepts, Chaitin machine computation, and Kolmogorov complexity. Step 7 introduces a finitary theory F; the details of the system F are not fully specified, but it needs to be able to formalize cut-elimination or normalization. Thus it seems that F can be taken to be, for instance, IΔ₀ + superexp or IΣ₁. The heart of the argument is reached in Step 16. Unfortunately, the argument becomes very uncertain here. Nelson argues that S* disproves a sequence of statements:
first \(A_{\kappa,0}\), then \(A_{\kappa,1}\), etc., up through \(A_{\kappa,I}\). The base case that \(S^*\) disproves \(A_{\kappa,0}\) is fine, but the later stages are unclear. It seems that the disproof of \(A_{\kappa,\delta}\) requires an assumption that \(S^*\) is consistent. The reason for this is that the Chaitin machine cannot be given the value of \(\delta\) as an input since the Kolmogorov complexity of \(\delta\) may not be sufficiently below that of \(\kappa\). The only alternative to explicitly specifying \(\delta\) that we can think of, is for the Chaitin machine to first search for the \(S^*\) proof that \(~A_{\kappa,\delta+1}\) and then also wait until \(\delta\) many strings are found to have Chaitin complexity less than \(\kappa\). This however assumes that \(S^*\) is consistent. Of course, \(S^*\) does not prove its own consistency. Perhaps Nelson had a different argument in mind, but this is our best attempt to flesh out his arguments. At any rate, Nelson was apparently aware of the potential problem here, since he earlier discusses the need for a system to prove the “consistency of its own arithmetization”.

In the spirit of a quote by Carl Sagan, “Extraordinary claims require extraordinary evidence”, Nelson planned to fulfill his inconsistency proof by exhibiting a fully formal, computer-verified derivation of a contradiction. That is, he planned not to prove that there is a proof of contradiction, but to actually exhibit an explicit proof of a contradiction. The first steps of this are carried out at the end of \textit{Elements}, and it is even further pursued in \textit{Balrog}. The \textit{Balrog} manuscript is still incomplete, as only six sections are complete, and at least ten sections were planned. The \textit{Balrog} manuscript is in essence a formalization of the “bootstrapping” of predicative arithmetic in the spirit of [5]. A remarkable feature of \textit{Balrog} is that proofs of theorems are indicated in a terse fashion that permits a Perl program, called \textit{qea}, to automatically verify the proofs. For instance, Theorems 13b. and 13i. of \textit{Balrog}, and their proofs, are typeset with the TeX code

\verbatim
\" \t//13b. 0 + 0 = 0 + 0 \n\sam13b. \"\p/13b. \r'H' \n/5 ; 0 + 0 \n\" \\
\" \t/13i. x + 0 = 0 + x \imp \r'S' x + 0 = 0 + \r'S' x \n\sam13i. \"\p/13i. \r'H' : x \n/12 ; \r'S' x \n/12 ; 0 ; x \n/12 ; x \n\" \\
\text{These indicate that 13b. is proved by substituting 0+0 for x in axiom a5., and that 13i. is proved by using definition r12. three times, first substituting Sx}\n\"
for $x$, then $0$ and $x$ for $x$ and $y$, and finally $x$ for $x$. After these substitutions, the desired conclusions follow propositionally from equality axioms. The qea system then automatically generated an expanded proof; the expanded proof was produced as a TeX file, and automatically converted to PDF. An example is shown in an appendix to the Balrog manuscript posted to the arXiv.

We of course believe that Peano arithmetic is consistent; thus we do not expect that Nelson’s project can be completed according to his plans. Nonetheless, there is much new in his papers that is of potential mathematical, philosophical and computational interest. For this reason, they are being posted to the arXiv. Two aspects of these papers seem particularly useful. The first aspect is the novel use of the “surprise examination” and Kolmogorov complexity; there is some possibility that similar techniques might lead to new separation results for fragments of arithmetic. The second aspect is Nelson’s automatic proof-checking via TeX and qea. This is highly interesting and provides a novel method of integrating human-readable proofs with computer verification of proofs.

The reader interested in further discussion of Nelson’s Predicative Arithmetic can consult the mostly-survey article [1]. The volume [2] contains papers about many other aspects of Nelson’s wide-ranging research. Other works by Nelson are available at math.princeton.edu/~nelson, including a number of philosophical works.


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August 31, 2015
Appendix

Example: Expanded proof of Balrog 13i

Nelson’s qea proof system consists of a Perl script which reads the TeX source code, and checks the proof correctness and generates an expanded version of the proof. The Balrog and Elements documents contained active hyperlinks, in a blue font, to these expanded proofs.

As an example, the expanded version of the proof of 13i. in Balrog as generated by qea is shown below.

Proof of Theorem 13i

The theorem to be proved is

\[ x + 0 = 0 + x \rightarrow Sx + 0 = 0 + Sx \]

Suppose the theorem does not hold. Then, with the variables held fixed,

\[
(H) \quad [[(x + 0) = (0 + x)] \& [\neg ((Sx) + 0) = (0 + (Sx))]]
\]

Special cases of the hypothesis and previous results:

0: \quad 0 + x = x + 0 \quad \text{from} \quad H:x
1: \quad \neg (Sx) + 0 = 0 + (Sx) \quad \text{from} \quad H:x
2: \quad (Sx) + 0 = Sx \quad \text{from} \quad 12;Sx
3: \quad S(0 + x) = 0 + (Sx) \quad \text{from} \quad 12;0;x
4: \quad x + 0 = x \quad \text{from} \quad 12; x

Equality substitutions:

5: \quad \neg 0 + x = x + 0 \quad \vee \quad \neg S(0 + x) = 0 + (Sx) \quad \vee \quad S(x + 0) = 0 + (Sx)
6: \quad \neg (Sx) + 0 = Sx \quad \vee \quad (Sx) + 0 = 0 + (Sx) \quad \vee \quad \neg Sx = 0 + (Sx)
7: \quad \neg x + 0 = x \quad \vee \quad \neg S(x + 0) = 0 + (Sx) \quad \vee \quad S(x) = 0 + (Sx)
Inferences:

8: \( \neg S(0 + x) = 0 + (Sx) \lor S(x + 0) = 0 + (Sx) \) by

0: \( 0 + x = x + 0 \)

5: \( \neg 0 + x = x + 0 \lor \neg S(0 + x) = 0 + (Sx) \lor S(x + 0) = 0 + (Sx) \)

9: \( \neg (Sx) + 0 = Sx \lor \neg 0 + (Sx) = Sx \) by

1: \( \neg (Sx) + 0 = 0 + (Sx) \)

6: \( \neg (Sx) + 0 = Sx \lor (Sx) + 0 = 0 + (Sx) \lor \neg 0 + (Sx) = Sx \)

10: \( \neg 0 + (Sx) = Sx \) by

2: \( (Sx) + 0 = Sx \)

9: \( \neg (Sx) + 0 = Sx \lor \neg 0 + (Sx) = Sx \)

11: \( S(x + 0) = 0 + (Sx) \) by

3: \( S(0 + x) = 0 + (Sx) \)

8: \( \neg S(0 + x) = 0 + (Sx) \lor S(x + 0) = 0 + (Sx) \)

12: \( \neg S(x + 0) = 0 + (Sx) \lor 0 + (Sx) = Sx \) by

4: \( x + 0 = x \)

7: \( \neg x + 0 = x \lor \neg S(x + 0) = 0 + (Sx) \lor 0 + (Sx) = Sx \)

13: \( \neg S(x + 0) = 0 + (Sx) \) by

10: \( \neg 0 + (Sx) = Sx \)

12: \( \neg S(x + 0) = 0 + (Sx) \lor 0 + (Sx) = Sx \)

14: \text{QEA} \text{ by}

11: \( S(x + 0) = 0 + (Sx) \)

13: \( \neg S(x + 0) = 0 + (Sx) \)