Lecture 19: Constructive Reals and Infinitesimal Calculus

Robert Constable

Abstract

We will present a very interesting theorem from Bishop and Bridges, and then we will briefly consider the nature of nonstandard models of the classical reals as presented in H.J. Keisler's book, *Elementary Calculus* and his free 203 page instructor's book entitled *Foundations of the Infinitesimal Calculus* [8] which is provided as a course resource with this lecture. We have not covered the basics of model theory on which the nonstandard models are based, and we will not attempt that because it is a subject not easily covered in constructive type theory.

1 Introduction

In the last lecture we discussed Bishop's approach to the constructive real numbers in terms of regular sequences of rational numbers. Some of the consequences of implementing this theory in Nuprl were discussed. Bishop's theory and its extensions by Bridges provide a *ground truth* for numerical analysis. Nuprl now offers an on-line service created by Dr. Bickford to compute the exact result of arithmetic operations on constructive real numbers. Later in the course we will discuss the latest Nuprl results in this area which includes incorporating Brouwer's *free choice sequences* and *continuity principle* (see Kleene and Vesley [9] and our recent Nuprl results [5]). In this lecture, we look at a constructive version of Cantor's theorem that the real numbers are not countable. This is proved in Bishop and Bridges, as Theorem (2.19) on page 27. The proof is quite illuminating, and was done in detail in the lecture following the proof on page 27.

It is interesting that one of Cantor's first results about the reals was geometric. He showed that it is possible to map a line segment ab one-to-one onto the points of the square whose sides are of length ab. This was a stunning result, but it is not a result of constructive analysis. Dr. Bickford's results from last week show a fully constructive theorem about the connectedness of lines in the plane that is also remarkable and constructively true.

2 The Constructive Reals are Uncountable

These notes will not repeat the proof from page 27 of Bishop and Bridges. It would be an interesting exercise to reformulate this theorem in a simpler way, just stating that given any *computable sequence* of real numbers (a_n) and given the closed interval of constructive reals in the interval $x_0 < y_0$, we can construct a real number x in this interval that is not equal to any of the reals a_n on the given list. Indeed, we can show that $x \neq a_n$ for all n. The definition of $x \neq a_n$ is very strong, saying that $x < a_n$ or $x > a_n$.

The lecture also discussed a number of the definitions and theorems leading up to Theorem 2.19. All of that material is covered in detail in pages 18 to 27.

In some sense this result might be very counter-intuitive and unexpected. If we think of the Nuprl implementation of the constructive reals, then each real is given by a "program" defining the algorithm for enumerating the sequences of converging rationals that constitute a real number. The sequences are given by Nuprl programs, and the programs are ultimately computed using Lisp functions. We can clear enumerate all such Lisp functions or even all of the Nuprl programs. So this is an argument that the constructive reals are actually countable!

What is wrong with this naive argument?

3 Nonstandard Constructive Reals

The calculus was developed by Leibniz and Newton using the notion of "infinitely small numbers" called *infinitesimals* nowadays. In the rigorous development of the calculus, we use so called "epsilon delta" arguments not infinitesimals. Here is the difference in the way we would define continuous functions on an interval $[a, b]$.

Definition 1: A function from reals to reals is *continuous* at x if and only if for every ϵ greater than 0, there is a real δ greater than 0, depending on ϵ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. We say that

$lim f(x) = f(c)$ as x goes to c.

Definition 2: A function from reals to reals is continuous at a point c if and only if whenever x is *infinitely close* to c, $f(x)$ is infinitely close to $f(c)$.

The second definition sounds very intuitive and is simpler, but what does the phrase "infinitely close" mean? If we have the idea of an infinitesimal around, as Leibniz did, then we know, the two values differ only by an infinitesimal.

So the question is, what is an infinitesimal? The subject of nonstandard analysis gives an answer by creating a model of the real numbers in which there are infinitely large numbers and infinitely small ones. The creation of these *nonstandard models* by Abraham Robinson [10] is regarded by H. J. Keisler as one of the greatest results in all of logic. It enabled him to write his 880 page book *Elementary Calculus* book [7] as well as his free *Foundations of Infinitesimal Calculus* [8] book that is included with this lecture.

Fundamental classical results in the subject called *model theory* led to a precise definition of the concept of an infinitesimal real number, the intuitive notion that Leibniz and Newton used in their development of the calculus. This subject relies heavily on nonconstructive results about logic that led to the subject called model theory. Prof Keisler is one of the leaders of this field. His textbook with C.C.Chang, *Model Theory* [6] is a classic, and it presents the results of Robinson needed for nonstandard analysis. Unfortunately these results depend on classical logic and advanced classical set theory. So it is difficult to incorporate them into constructive type theory. Several people have explored this path. However it does seem possible to give a plausible constructive account of infinitesimals, and we will look at this topic in one of the next three lectures.

These results also depend on classical first-order logic. This is a more tractable notion to explain constructively. So we will examine this topic briefly and also discuss the differences between classical set theory and type theory. It is very interesting that Aczel has given a constructive account of set theory in type theory in a series of accessible articles [1, 2, 3, 4].

References

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