

APPENDIX B.

THE DOCTRINE OF TYPES.

497. THE doctrine of types is here put forward tentatively, as affording a possible solution of the contradiction; but it requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties. In case, however, it should be found to be a first step towards the truth, I shall endeavour in this Appendix to set forth its main outlines, as well as some problems which it fails to solve.

Every propositional function $\phi(x)$ —so it is contended—has, in addition to its range of truth, a range of significance, *i.e.* a range within which x must lie if $\phi(x)$ is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second point is that ranges of significance form *types*, *i.e.* if x belongs to the range of significance of $\phi(x)$, then there is a class of objects, the *type* of x , all of which must also belong to the range of significance of $\phi(x)$, however ϕ may be varied; and the range of significance is always either a single type or a sum of several whole types. The second point is less precise than the first, and the case of numbers introduces difficulties; but in what follows its importance and meaning will, I hope, become plainer.

A *term* or *individual* is any object which is not a range. This is the lowest type of object. If such an object—say a certain point in space—occurs in a proposition, any other individual may *always* be substituted without loss of significance. What we called, in Chapter VI, the class as one, is an individual, provided its members are individuals: the objects of daily life, persons, tables, chairs, apples, etc., are classes as one. (A person is a class of psychical existents, the others are classes of material points, with perhaps some reference to secondary qualities.) These objects, therefore, are of the same type as simple individuals. It would seem that all objects designated by single words, whether things or concepts, are of this type. Thus *e.g.* the relations that occur in actual relational propositions are of the same type as things, though relations in extension, which are what Symbolic Logic employs, are of a different type. (The intensional relations which occur in ordinary relational propositions are not determinate when their extensions are given, but the extensional relations of Symbolic Logic are classes of couples.) Individuals are the only objects of which numbers cannot be significantly asserted.

The next type consists of ranges or classes of individuals. (No ordinal ideas are to be associated with the word *range*.) Thus "Brown and Jones" is an object of this type, and will in general not yield a significant proposition if substituted for "Brown" in any true or false proposition of which Brown is a constituent. (This constitutes, in a kind of way, a justification for the grammatical distinction of singular and plural; but the analogy is not close, since a range may have one term or more, and where it has many, it may yet appear as singular in certain propositions.) If u be a range determined by a propositional function $\phi(x)$, not- u will consist of all objects for which $\phi(x)$ is false, so that not- u is contained in the range of significance of $\phi(x)$, and contains only objects of the same type as the members of u . There is a difficulty in this connection, arising from the fact that two propositional functions $\phi(x)$, $\psi(x)$ may have the same range of truth u , while their ranges of significance may be different; thus not- u becomes ambiguous. There will always be a minimum type within which u is contained, and not- u may be defined as the rest of this type. (The sum of two or more types is a type; a minimum type is one which is not such a sum.) In view of the Contradiction, this view seems the best; for not- u must be the range of falsehood of " x is a u ," and " x is an x " must be in general meaningless; consequently " x is a u " must require that x and u should be of different types. It is doubtful whether this result can be insured except by confining ourselves, in this connection, to minimum types.

There is an unavoidable conflict with common sense in the necessity for denying that a mixed class (*i.e.* one whose members are not all of the same minimum type) can ever be of the same type as one of its members. Consider, for example, such phrases as "Heine and the French." If this is to be a class consisting of two individuals, "the French" must be understood as "the French nation," *i.e.*, as the class as one. If we are speaking of the French as many, we get a class consisting not of two members, but of one more than there are Frenchmen. Whether it is possible to form a class of which one member is Heine, while the other is the French as many, is a point to which I shall return later; for the present it is enough to remark that, if there be such a class, it must, if the Contradiction is to be avoided, be of a different type both from classes of individuals and from classes of classes of individuals.

The next type after classes of individuals consists of classes of classes of individuals. Such are, for example, associations of clubs; the members of such associations, the clubs, are themselves classes of individuals. It will be convenient to speak of *classes* only where we have classes of individuals, of *classes of classes* only where we have classes of classes of individuals, and so on. For the general notion, I shall use the word *range*. There is a progression of such types, since a range may be formed of objects of any given type, and the result is a range of higher type than its members.

A new series of types begins with the couple with sense. A range of such types is what Symbolic Logic treats as a relation: this is the extensional view of relations. We may then form ranges of relations, or relations of relations, or relations of couples (such as separation in Projective Geometry*),

* Cf. § 203.

or relations of individuals to couples, and so on; and in this way we get, not merely a single progression, but a whole infinite series of progressions. We have also the types formed of trios, which are the members of triple relations taken in extension as ranges; but of trios there are several kinds that are reducible to previous types. Thus if $\phi(x, y, z)$ be a propositional function, it may be a product of propositions $\phi_1(x) \cdot \phi_2(y) \cdot \phi_3(z)$ or a product $\phi_1(x) \cdot \phi_2(y, z)$, or a proposition about x and the couple (y, z) , or it may be analyzable in other analogous ways. In such cases, a new type does not arise. But if our proposition is not so analyzable—and there seems no *a priori* reason why it should always be so—then we obtain a new type, namely the trio. We can form ranges of trios, couples of trios, trios of trios, couples of a trio and an individual, and so on. All these yield new types. Thus we obtain an immense hierarchy of types, and it is difficult to be sure how many there may be; but the method of obtaining new types suggests that the total number is only α_0 (the number of finite integers), since the series obtained more or less resembles the series of rationals in the order $1, 2, \dots, n, \dots, 1/2, 1/3, \dots, 1/n, \dots, 2/3, \dots, 2/5, \dots, 2/(2n+1), \dots$. This, however, is only a conjecture.

Each of the types above enumerated is a *minimum* type; *i.e.*, if $\phi(x)$ be a propositional function which is significant for one value of x belonging to one of the above types, then $\phi(x)$ is significant for every value of x belonging to the said type. But it would seem—though of this I am doubtful—that the sum of any number of minimum types is a type, *i.e.* is a range of significance for certain propositional functions. Whether or not this is universally true, *all ranges* certainly form a type, since every range has a number; and so do all objects, since every object is identical with itself.

Outside the above series of types lies the type *proposition*; and from this as starting-point a new hierarchy, one might suppose, could be started; but there are certain difficulties in the way of such a view, which render it doubtful whether propositions can be treated like other objects.

498. Numbers, also, are a type lying outside the above series, and presenting certain difficulties, owing to the fact that every number selects certain objects out of every other type of ranges, namely those ranges which have the given number of members. This renders the obvious definition of 0 erroneous; for every type of range will have its own null-range, which will be a member of 0 considered as a range of ranges, so that we cannot say that 0 is the range whose only member is *the* null-range. Also numbers require a consideration of the totality of types and ranges; and in this consideration there may be difficulties.

Since all ranges have numbers, ranges are a range; consequently $x \epsilon x$ is sometimes significant, and in these cases its denial is also significant. Consequently there is a range w of ranges for which $x \epsilon x$ is false: thus the Contradiction proves that this range w does not belong to the range of significance of $x \epsilon x$. We may observe that $x \epsilon x$ can only be significant when x is of a type of infinite order, since, in $x \epsilon u$, u must always be of a type higher by one than x ; but the range of all ranges is of course of a type of infinite order.

Since numbers are a type, the propositional function “ x is not a u ,”

where u is a range of numbers, must mean " x is a number which is not a u "; unless, indeed, to escape this somewhat paradoxical result, we say that, although numbers are a type in regard to certain propositions, they are not a type in regard to such propositions as " u is contained in v " or " x is a u ." Such a view is perfectly tenable, though it leads to complications of which it is hard to see the end.

That propositions are a type results from the fact—if it be a fact—that only propositions can significantly be said to be true or false. Certainly true propositions appear to form a type, since they alone are asserted (cf. Appendix A. § 479). But if so, the number of propositions is as great as that of all objects absolutely, since every object is identical with itself, and " x is identical with x " has a one-one relation to x . In this there are, however, two difficulties. First, what we called the propositional concept appears to be always an individual; consequently there should be no more propositions than individuals. Secondly, if it is possible, as it seems to be, to form ranges of propositions, there must be more such ranges than there are propositions, although such ranges are only some among objects (cf. § 343). These two difficulties are very serious, and demand a full discussion.

499. The first point may be illustrated by somewhat simpler ones. There are, we know, more classes than individuals; but predicates are individuals. Consequently not all classes have defining predicates. This result, which is also deducible from the Contradiction, shows how necessary it is to distinguish classes from predicates, and to adhere to the extensional view of classes. Similarly there are more ranges of couples than there are couples, and therefore more than there are individuals; but verbs, which express relations intensionally, are individuals. Consequently not every range of couples forms the extension of some verb, although every such range forms the extension of some propositional function containing two variables. Although, therefore, verbs are essential in the logical genesis of such propositional functions, the intensional standpoint is inadequate to give all the objects which Symbolic Logic regards as relations.

In the case of propositions, it seems as though there were always an associated verbal noun which is an individual. We have " x is identical with x " and "the self-identity of x ," " x differs from y " and "the difference of x and y "; and so on. The verbal noun, which is what we called the propositional concept, appears on inspection to be an individual; but this is impossible, for "the self-identity of x " has as many values as there are objects, and therefore more values than there are individuals. This results from the fact that there are propositions concerning every conceivable object, and the definition of identity shows (§ 26) that every object concerning which there are propositions, is identical with itself. The only method of evading this difficulty is to deny that propositional concepts are individuals; and this seems to be the course to which we are driven. It is undeniable, however, that a propositional concept and a colour are two objects; hence we shall have to admit that it is possible to form mixed ranges, whose members are not all of the same type; but such ranges will be always of a different type from what we may call pure ranges, *i.e.* such as have only members of one type. The propositional concept seems, in fact, to be nothing

other than the proposition itself, the difference being merely the psychological one that *we* do not assert the proposition in the one case, and do assert it in the other.

500. The second point presents greater difficulties. We cannot deny that there are ranges of propositions, for we often wish to assert the logical product of such ranges; yet we cannot admit that there are more ranges than propositions. At first sight, the difficulty might be thought to be solved by the fact that there is a proposition associated with every range of propositions which is not null, namely the logical product of the propositions of the range*; but this does not destroy Cantor's proof that a range has more sub-ranges than members. Let us apply the proof by assuming a particular one-one relation, which associates every proposition p which is not a logical product with the range whose only member is p , while it associates the product of all propositions with the null-range of propositions, and associates every other logical product of propositions with the range of its own factors. Then the range w which, by the general principle of Cantor's proof, is not correlated with any proposition, is the range of propositions which are logical products, but are not themselves factors of themselves. But, by the definition of the correlating relation, w ought to be correlated with the logical product of w . It will be found that the old contradiction breaks out afresh; for we can prove that the logical product of w both is and is not a member of w . This seems to show that there is no such range as w ; but the doctrine of types does not show why there is no such range. It seems to follow that the Contradiction requires further subtleties for its solution; but what these are, I am at a loss to imagine.

Let us state this new contradiction more fully. If m be a class of propositions, the proposition "every m is true" may or may not be itself an m . But there is a one-one relation of this proposition to m : if n be different from m , "every n is true" is not the same proposition as "every m is true." Consider now the whole class of propositions of the form "every m is true," and having the property of not being members of their respective m 's. Let this class be w , and let p be the proposition "every w is true." If p is a w , it must possess the defining property of w ; but this property demands that p should not be a w . On the other hand, if p be not a w , then p does possess the defining property of w , and therefore is a w . Thus the contradiction appears unavoidable.

In order to deal with this contradiction, it is desirable to reopen the question of the identity of equivalent propositional functions and of the nature of the logical product of two propositions. These questions arise as follows. If m be a class of propositions, their logical product is the proposition "every m is true," which I shall denote by $\wedge m$. If we now consider the logical product of the class of propositions composed of m

* It might be doubted whether the relation of ranges of propositions to their logical products is one-one or many-one. For example, does the logical product of p and q and r differ from that of pq and r ? A reference to the definition of the logical product (p. 21) will set this doubt at rest; for the two logical products in question, though equivalent, are by no means identical. Consequently there is a one-one relation of all ranges of propositions to some propositions, which is directly contradictory to Cantor's theorem.

together with $\wedge'm$, this is equivalent to "Every m is true and every m is true,' *i.e.* to "every m is true" *i.e.* to $\wedge'm$. Thus the logical product of the new class of propositions is equivalent to a member of the new class, which is the same as the logical product of m . Thus if we identify equivalent propositional functions ($\wedge'm$ being a propositional function of m), the proof of the above contradiction fails, since every proposition of the form $\wedge'm$ is the logical product both of a class of which it is a member and of a class of which it is not a member.

But such an escape is, in reality, impracticable, for it is quite self-evident that equivalent propositional functions are often not identical. Who will maintain, for example, that " x is an even prime other than 2" is identical with " x is one of Charles II.'s wise deeds or foolish sayings"? Yet these are equivalent, if a well-known epitaph is to be credited. The logical product of all the propositions of the class composed of m and $\wedge'm$ is "Every proposition which either is an m or asserts that every m is true, is true"; and this is not identical with "every m is true," although the two are equivalent. Thus there seems no simple method of avoiding the contradiction in question.

The close analogy of this contradiction with the one discussed in Chapter x strongly suggests that the two must have the same solution, or at least very similar solutions. It is possible, of course, to hold that propositions themselves are of various types, and that logical products must have propositions of only one type as factors. But this suggestion seems harsh and highly artificial.

To sum up: it appears that the special contradiction of Chapter x is solved by the doctrine of types, but that there is at least one closely analogous contradiction which is probably not soluble by this doctrine. The totality of all logical objects, or of all propositions, involves, it would seem, a fundamental logical difficulty. What the complete solution of the difficulty may be, I have not succeeded in discovering; but as it affects the very foundations of reasoning, I earnestly commend the study of it to the attention of all students of logic.

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