Induction and Coinduction

Fixpoint Types

$F : \text{Type} \to \text{Type}$

Example

$FX = 1 + \mathbb{Z} \times X$

$T : \text{Type}$ iso: $FT \simeq T$

$1 + \mathbb{Z} \times T \simeq T$

least → $List(\mathbb{Z})$

Greatest ↙ $InfList(\mathbb{Z})$

$CoList(\mathbb{Z})$

Inductive Types : F is a functor

$$\begin{tikzcd} F\mu F \arrow[r, "F(fold(alg))"] \arrow[d, "in"] & FX \arrow[d, "alg"] \\ \mu F \arrow[r, dashed, "fold(alg)"'] & X \end{tikzcd}$$

↑ algebra of F

$$\begin{tikzcd} 1+\mathbb{Z}\times L(\mathbb{Z}) \arrow[r, "1+\mathbb{Z}\times(fold\ 0\ +)"] \arrow[d, "{[nil, cons]}"] & 1+\mathbb{Z}\times\mathbb{R} \arrow[d, "{[0,+]}"] \\ L(\mathbb{Z}) \arrow[r, "fold\ 0\ +"'] & \mathbb{R} \end{tikzcd}$$

$fold\ 0 + nil = 0$

$fold\ 0 + (cons\ n\ \ell) = n + fold\ 0 + \ell$

Coinductive Types

$$\begin{tikzcd} X \arrow[r, dashed, "cofold(coalg)"] \arrow[d, "coalg"] & \nu F \arrow[d, "out"] \\ FX \arrow[r, "F(cofold(coalg))"'] & F\nu F \end{tikzcd}$$

↑ coalgebra of F

$$\begin{tikzcd} {[0,1)} \arrow[r, "cofold(firstdigit)"] \arrow[d, "firstdigit"] & I(\mathbb{Z}) \arrow[d] \\ 1+\mathbb{Z}\times[0,1) \arrow[r] & 1+\mathbb{Z}\times I(\mathbb{Z}) \end{tikzcd}$$

Structural Recursion

$fix\ f\ \overset{\mu F}{i}\ \overset{args}{\vdots} =$

$e(f, i\ \overset{args}{})$

↑ every argument to f must be a subexpression of i (recursive inputs must become strictly smaller)

guarantees termination (after construction of the input)

Corecursion

cofix f args : νF

$= e(f, \text{args})$

every output of f must be applied to a constructor (recursive outputs must become strictly larger)

guarantees termination (after definition of the output)

Well-Founded Recursion

wfix f args

$= e(f, \text{args})$

inputs to f are smaller than args for some definition of smaller

Well-founded Relation on T is

$R : T \to T \to \text{Prop}$

and a proof of $\forall t.\ \text{Acc}\ R\ t$

where

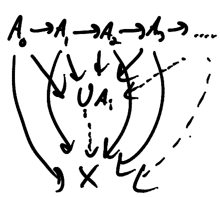
Inductive Acc R : $T \to \text{Prop}$

acc_intro $(t:T) : (\forall t'.\ t'\,R\,t \to \text{Acc}_R\ t') \to \text{Acc}\ R\ t$

Building Inductive Types

Initial Object (called 0) has a unique morphism i_A to any object A

ω-colimits



Suppose F preserves ω-colimits: $F(\cup A_i) \cong \cup(FA_i)$