Lifting

Vector Notation

$\vec{v} + S$ or $\vec{x} + \vec{y}$

Vectors of length n as a Functor

$-^n : Set \to Set$

$X^n = X \times \ldots \times X$ ← n times

f^n: $f \downarrow \ \ldots \ \downarrow f$

$Y \times \ldots \times Y$

Combining Pairs of Vectors of length n

If I have a vector $\vec{x}$ of type X^n

and I have another vector $\vec{y}$ of type Y^n

$X^n \times Y^n \longrightarrow (X \times Y)^n$

by combining component-wise pairs

This is called a Monoidal Functor on Set

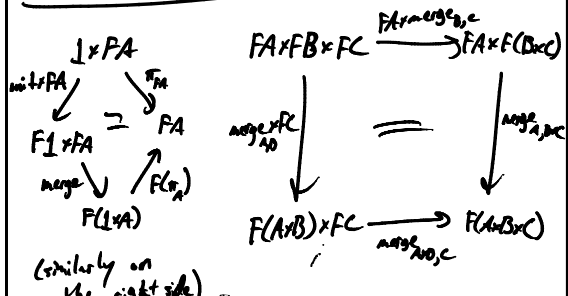
$F : Set \to Set$

$merge_{A,B} : FA \times FB \to F(A \times B)$

$unit : 1 \to F1$

with properties...

Monoidal Laws



Combining Expressions

$$\frac{[\![\Gamma \vdash e : F\tau]\!] = f \quad [\![\Gamma \vdash e' : F\tau']\!] = f' \quad op : \tau \times \tau' \to \tau''}{[\![\Gamma \vdash e \ op \ e' : F\tau'']\!] = \lambda G.\, F(op)(merge_{\tau,\tau'}(fG, f'G))}$$

works great for vectors of the same length

but what about heterogeneous vectors? (i.e. lists)

List as a Monoidal Functor

$merge_{A,B} : List\,A \times List\,B \to List(A \times B)$

= zip (ignoring extra elements)

$unit : 1 \to List\,1$

= ~~[()]~~ [(), (), (), ……]

How to work with single lists?

Lack of unit implies what?

$\Gamma \vdash e : \tau \not\Rightarrow \Gamma \vdash e : F\tau$

This means op uniformity can't treat everything as an F

so let's embrace chaos!

Monoidal Family of Functors

A monoid ⟨Eff, ε, ·⟩

A map $F : Eff \to (Set \to Set)$

A family of nat trans $merge_{\epsilon,\epsilon'} : F_\epsilon A \times F_{\epsilon'} B \to F_{\epsilon\cdot\epsilon'}(A \times B)$

A trans $unit : 1 \to F_\epsilon 1$

satisfying properties analogous to monoidal laws

Example:

$\mathbb{N}_\infty$, ∞, min

λn. —"

zip

[(),(),(), ……]

So far

		e.g.
Functor:	one lifted expression	$1 + \vec{v}$
Monoidal:	multiple adjacent lifted expressions	$\vec{x} + \vec{y}$
Monad:	multiple nested lifted expressions	$\overrightarrow{\vec{x}.\text{factors}()}$

but there are subtleties

Other Monoidal Structures for List

ltor: $List\,A \times List\,B \to List(A \times B)$

(as, bs) ↦ as.map(λa. bs.map(λb.⟨a,b⟩)).flatten()

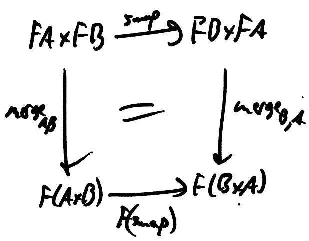
[0,1], [2,3,4] ↦ [⟨0,2⟩, ⟨0,3⟩, ⟨0,4⟩, ⟨1,2⟩, ⟨1,3⟩, ⟨1,4⟩]

rtol: $List\,A \times List\,B \to List(A \times B)$

(as, bs) ↦ bs.map(λb. as.map(λa.⟨a,b⟩)).flatten()

[0,1], [2,3,4] ↦ [⟨0,2⟩, ⟨1,2⟩, ⟨0,3⟩, ⟨1,3⟩, ⟨0,4⟩, ⟨1,4⟩]

Symmetric Monoidal Functors



zip ∋ symmetric

ltor & rtol
are not symmetric