Quotient / Types

What is a quotient type?

T/R

T : Type

$R : T \to T \to \mathrm{Prop}$

and forms an equivalence relation

What do you do with it?

$$[-]_R : T \to T/R$$

$$\frac{\Gamma, t:T \vdash e:A \qquad \Gamma \vdash p : \forall t,t':T.\ t\,R\,t' \to e[t] = e[t']}{\Gamma, q : T/R \vdash \text{choose } t \text{ from } q \text{ in } e \text{ using } p : A}$$

Extensionality

$$(\forall a.\ f\,a = g\,a) \Rightarrow f = g$$

Extensionality via Quotients

$$A \to B/_{ext} \longrightarrow A \to B$$

$$\frac{\vdash \lambda f.\lambda g.\lambda e.\,e\,a : \forall f,g.\ ext\,f\,g \to f\,a = g\,a}{q : A \to B/_{ext} \vdash \lambda a.\ \text{choose } f \text{ from } q \text{ in } f\,a \text{ using } \lambda f.\lambda g.\lambda e.\,e\,a}$$

$[f]_{ext}$

Quotients in Practice

$$\mathbb{Z} \times \mathbb{N}_+ / _{\lambda(n,d).\,\lambda(n',d').\,nd' = n'd} = \mathbb{Q}$$

$\mathbb{N} \to \mathbb{Q}$ that ~~converge~~ eventually coconverge $= \mathbb{R}$

Canonicalization

$$\mathrm{rep} : T/R \xrightarrow{} T \qquad [-]_R : T \to T/R$$

$s;r = id$

$r \circ s = id$

$\forall q.\ [\mathrm{rep}\ q]_R = q$

Axiom of Choice:

All quotient types have a canonicalizer.

All $[-]_R$ have a section.

Proof Irrelevance vs. Proof Erasure

Proof Irrelevance: all proofs used lead to same result

$$\mathrm{Proof} \to T$$

Proof Erasure: proof is not actually needed for implementation

Example of Difference

$$f : \forall P.\ \mathrm{Proof}(P \vee \neg P) \longrightarrow \mathbb{B}$$

f predicates truth of P

$$\cancel{\frac{x \to 0 + x}{T}} \to \frac{x \to 0}{T} + \frac{x}{T}$$

Constructive Choice

$$\forall A.\ \mathrm{dec}(=_A) \to \forall B, R.\ (A \to B/R) \longrightarrow \frac{A \to B}{A \to R}$$

to get $A \to B$, build it dynamically by matching its inputs

suppose an input "a" comes along

first compare with all prior inputs, if so use same output

otherwise use $f\,a$ to get a $[b]_R$ and output b