Dominion Theory

Fixpoint Recursion

$\text{fix } x{:}\tau.\ e^{:\tau} {}^{:\tau} = e[x \mapsto \text{fix } x.e]$

$\text{fix } e^{:\tau \to \tau} \ : \tau$

$e\,(\text{fix } e) = \text{fix}'e$

Domain – ωCPO

X: Set

$\leq$: Partial-Order X

$\bigsqcup_\omega$: Takes an infinite increasing sequence and returns a least upper bound of that sequence

$x_0 \leq x_1 \leq x_2 \leq x_3 \cdots$

$\bigsqcup_{i \in \omega} x_i$

$\omega CPO_\perp$

$\omega CPO\ \langle X, \leq, \bigsqcup_\omega \rangle$

$\perp : X$

such that

$\perp \leq x$ for all x

ωCPO morphism

$f : \langle X, \leq, \bigsqcup_\omega \rangle \longrightarrow \langle Y, \leq, \bigsqcup_\omega \rangle$

$f : X \to Y$

such that $x \leq x' \Rightarrow f(x) \leq f(x')$

and $f(\bigsqcup_{i \in \omega} x_i) = \bigsqcup_{i \in \omega} f(x_i)$

$\omega CPO_\perp$ morphism $f(\perp_X) = \perp_Y$

fix using $\omega CPO_{(\perp)}$

$\langle X, \leq, \bigsqcup_\omega, \perp \rangle$ is an $\omega CPO_\perp$

$f : \langle X, \leq, \bigsqcup_\omega \rangle \to \langle X, \leq, \bigsqcup_\omega \rangle$ is an ωCPO morphism (w/ ⊥)

$x_0 = \perp$
$x_{i+1} = f(x_i)$ is an ω-chain

$\text{fix } f = \bigsqcup_{i \in \omega} x_i$

$f(\text{fix } f) = f(\bigsqcup_{i \in \omega} x_i) = \bigsqcup_{i \in \omega} f(x_i) = \bigsqcup_{i \in \omega} f(f^i(\perp)) = \bigsqcup_{i \in \omega} f^{i+1}(\perp) = \bigsqcup_{i \in \omega} f^i(\perp)$

Given poset $(X, \leq)$, define Stream(X)

CoInductive Stream X : Type
  := next : $X \to$ Stream $X \to$ Stream X.

CoFixpoint nats (n:Nat) : Stream Nat
  := next n (nats (S n)).

CoInductive $\leq_s$: Stream $X \to$ Stream $X \to$ Prop
  := next$_\leq$ $(x\ x' : X)$ $(s\ s' :$ Stream $X)$
    : $x \leq x' \to s \leq_s s' \to$ next $x\ s \leq_s$ next $x'\ s'$

Increasing Streams

$(x_i)_{i \in \mathbb{N}}$ where $x_i \leq x_{i+1}$

increasing stream

$\left((x_{i,j})_{i \in \mathbb{N}}\right)_{j \in \mathbb{N}}$ increasing stream of increasing streams

ideally we could use $(x_{k,k})_{k \in \mathbb{N}}$

this is an increasing stream
but it's not bigger than the rest!

Solution: use eventually smaller than