Monads and Comonads

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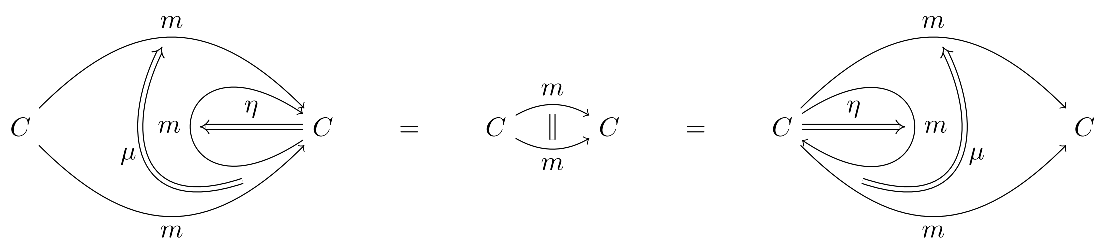
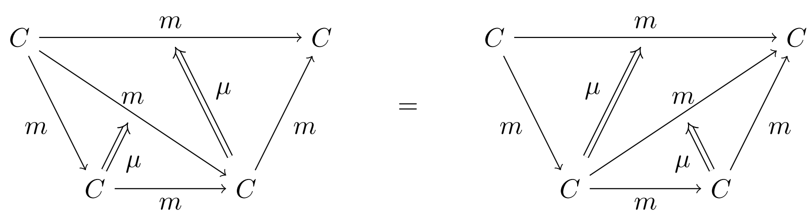
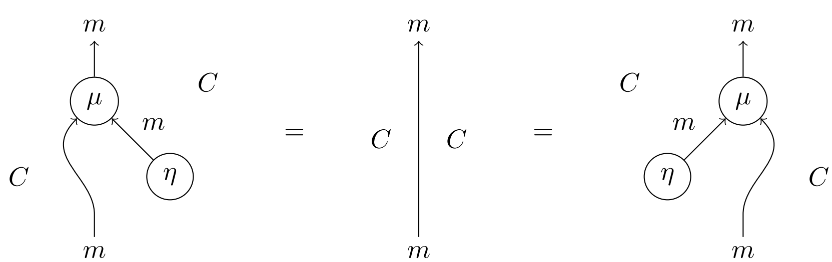
Definition (Monad)**.** A monad in a given 2-category is comprised of the following:

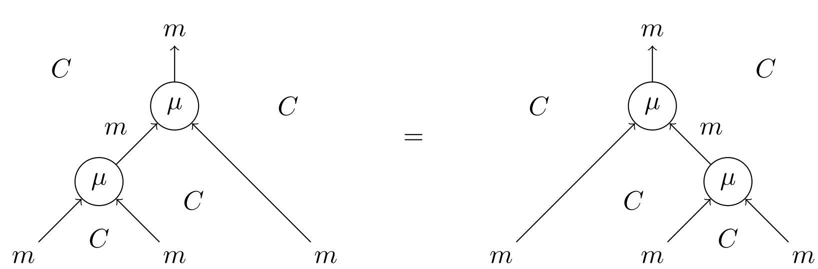
- 0-cell C
- 1-cell $m : C \to C$ (generally referred to as the monad)
- 2-cells $\eta : id_C \Rightarrow m : C \Rightarrow C$ (called the unit) and $\mu : m\,;m \Rightarrow m : C \to C$ (called the join)
- such that the following identity and associativity laws hold:





Remark. In terms of string diagrams, the identity and associative laws are formulated as follows:





Remark. Given a 2-category, one can construct a multicategory whose objects are the 1-cells of the multicategory and whose morphisms are 2-cells from the composition of the inputs to the output. A monad is an internal monoid of that multicategory.

Theorem. *For any monad $\langle C, m, \eta, \mu\rangle$ and $n : \mathbb{N}$, all 2-cells from m^n to m built from η, μ, and identities are equal.*

Example. The functor $\mathbb{O}$ serves as a basis for a monad on **Set** in the 2-category **Set**. The unit is some, and the join $\mu_A : \mathbb{O}(\mathbb{O}(A)) \to \mathbb{O}(A)$ maps $\mathsf{some}(\mathsf{some}(a))$ to $\mathsf{some}(a)$ and maps $\mathsf{some}(\mathsf{none})$ and none to none.

Example. $\langle \mathbf{Set}, \mathbb{L}, \mathtt{singleton}, \mathtt{flatten} \rangle$ is a monad in **Cat**. Similarly, $\langle \mathbf{Set}, \mathbb{P}, \lambda x.\, \{x\}, \bigcup \rangle$ is also a monad in **Set**. There is also a monad for the functor $\mathbb{M} : \mathbf{Set} \to \mathbf{Set}$ that maps sets A to the set of finite multisets/bags of A, i.e. finite collections of A elements in which duplicates matter but order does not. And there is a monad for the functor $\mathbb{F} : \mathbf{Set} \to \mathbf{Set}$ that maps sets A to the set of finite subsets of A.

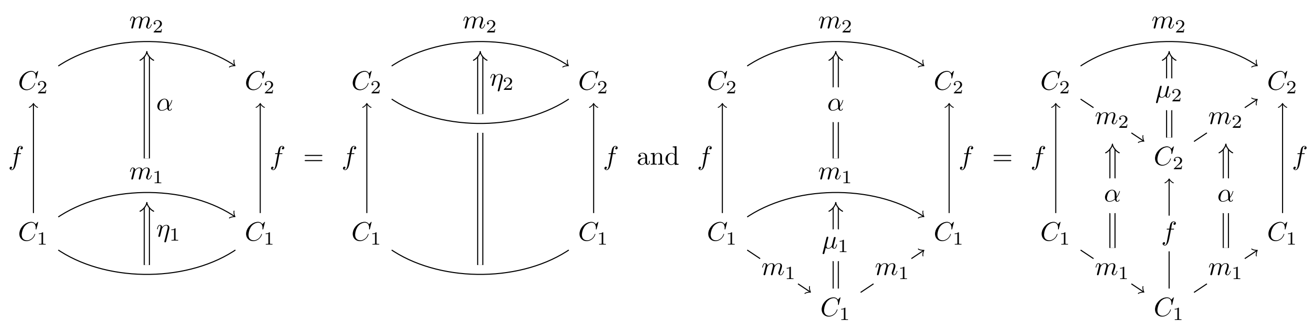
Example. Given a set C, the functor $C \to \cdot : \mathbf{Set} \to \mathbf{Set}$ is a monad. The unit is the natural transformation mapping $a \in A$ to $(\lambda c \in C.\, a) \in C \to A$. The join is the natural transformation mapping $f \in C \to (C \to A)$ to $(\lambda c \in C.\, f(c)(c)) \in C \to A$.

Example. Given a set S, the functor $S \to S \times \cdot : \mathbf{Set} \to \mathbf{Set}$ is a monad. The unit is the natural transformation mapping $a \in A$ to $(\lambda s \in S.\, \langle s, a \rangle) \in S \to S \times A$. The join is the natural transformation mapping $f \in S \to (S \times (S \to S \times A))$ to $(\lambda s.\, \pi_2(f(s))(\pi_1(f(s)))) \in S \to S \times A$.

Example. Given a monoid $\langle M, e, * \rangle$, the functor $M \times \cdot : \mathbf{Set} \to \mathbf{Set}$ is a monad. The unit is the natural transformation mapping $a \in A$ to $\langle e, a \rangle \in M \times A$. The join is the natural transformation mapping $\langle m, \langle m', a \rangle \rangle \in M \times (M \times A)$ to $\langle m * m', a \rangle \in M \times A$.

Example. Given a graph $\langle V, E, s, t \rangle$ one can define the set of paths as alternating lists of vertices and edges $(v_0, e_0, v_1, e_1, \ldots, v_n)$ with the property that $s(e_i) = v_i$ and $t(e_i) = v_{i+1}$ for all indices i. The source of such a path is v_0, and the target is v_n. Thus we have a graph $\langle V, \mathrm{Path}(E), s_{\mathrm{Path}}, t_{\mathrm{Path}} \rangle$. This Path construction extends to a monad. For both the unit and join, the function on vertices on simply the identity. As for edges, the unit maps an edge e to the path $(s(e), e, t(e))$, and the join essentially flattens paths of paths.

Definition (Monad Morphism)**.** An (oplax) monad morphism from $\langle C_1, m_1, \eta_1, \mu_1 \rangle$ to $\langle C_2, m_2, \eta_2, \mu_2 \rangle$ is a 1-cell $f : C_1 \to C_2$ and a 2-cell $\alpha : m_1 \,;\, f \Rightarrow f \,;\, m_2$ such that



Example. The obvious natural transformations from $\mathbb{L}$ to $\mathbb{M}$ to $\mathbb{F}$ to $\mathbb{P}$ are all monad morphisms where the 1-cell f is the identity functor of **Set**.

Example. The functor $\pi_E : \mathbf{Graph} \to \mathbf{Set}$ has the property that $\mathrm{Path} \,;\, \pi_E$ equals $\pi_E \,;\, \mathbb{L}$. This 1-cell π_E in fact forms a monad morphism from Path to $\mathbb{L}$ where the 2-cell is the identity 2-cell.

Definition (Comonad)**.** A comonad in a 2-category is a 0-cell C, a 1-cell $c : C \to C$ (typically called the comonad), and 2-cells $\varepsilon : m \Rightarrow id_C$ (often called the counit) and $\delta : m \Rightarrow m \,;\, m$ (often called the cojoin or comultiplication) satisfying equalities dual to the identity and associativity laws of monads.

Example. Given any set C, the functor $C \times \cdot : \mathbf{Set} \to \mathbf{Set}$ is a comonad on **Set** in **Cat**. The counit is the natural transformation mapping $\langle c, a \rangle \in C \times A$ to $a \in A$. The cojoin is the natural transformation mapping $\langle c, a \rangle \in C \times A$ to $\langle c, \langle c, a \rangle \rangle \in C \times (C \times A)$.

Example. Given a monoid $\langle M, e, * \rangle$, the functor $M \to \cdot : \mathbf{Set} \to \mathbf{Set}$ is a comonad. The counit is the natural transformation mapping $f \in M \to A$ to $f(e) \in A$. The cojoin is the natural transformation mapping $f \in M \to A$ to $\lambda m \in M.\, \lambda m' \in M.\, s(m * m') \in M \to (M \to A)$. When the monoid is $\langle \mathbb{N}, 0, + \rangle$, this is known as the stream comonad.

Definition (Comonad Morphism)**.** Just as a comonad in a 2-category $\mathbf{C}$ coincides with a monad in $\mathbf{C}^{\mathrm{co}}$, a comonad morphism in $\mathbf{C}$ coincides with a monad morphism in $\mathbf{C}^{\mathrm{co}}$.