Exercise 1. singleton: Set ⇒ L and flatten: LL ⇒ L are two particularly important natural transformations pertaining to lists. The following equate various compositions of these natural transformations: Write each of the diagrams as polymorphic programs of the form \( \lambda l : \tau. \ldots \) for some type \( \tau \) (referencing a type variable \( \alpha \)) using the following “library” functions with explicit subscripts (i.e. use explicit type arguments):

\[
\begin{align*}
\text{map}_{\alpha,\beta} &: (\alpha \to \beta) \to (L\alpha \to L\beta) \\
\text{singleton}_{\alpha} &: \alpha \to L\alpha \\
\text{flatten}_{\alpha} &: LL\alpha \to L\alpha
\end{align*}
\]

Then prove the above equalities.

Exercise 2. Let \( \text{Map}(C, D) \) be the following relational category:

- an object \( V \) maps each object \( A \) of \( C \) to an object \( V(A) \) of \( D \)
- a morphism \( E \) from \( V_1 \) to \( V_2 \) maps each morphism \( m : A \to B \) of \( C \) to a morphism \( E(m) : V_1(A) \to V_2(B) \)
- a (possibly empty) path \( V_0 \xrightarrow{E_1} V_1 \xrightarrow{} V_{n-1} \xrightarrow{E_n} V_n \) composes to \( V_0 \xrightarrow{E} V_n \) whenever

\[
\forall A_0 \xrightarrow{m_0} A_1 \xrightarrow{} A_{n-1} \xrightarrow{m_n} A_n, \quad E_1(m_1); \ldots; E_n(m_n) = E(m_1; \ldots; m_n)
\]

This is a relational category because composition is defined as a relation, rather than function, from paths to morphisms (that satisfies laws akin to associative/identity).

Prove that functors from \( C \) to \( D \) precisely coincide with endomorphisms \( E : V \to V \) of \( \text{Map}(C, D) \) with the property that all paths of the form \( E^* : V \to V \), i.e. arbitrary repetitions of \( E \), compose to \( E \). Prove that natural transformations from the functor coinciding with \( V_1 \xrightarrow{E_1} V_1 \) to the functor coinciding with \( V_2 \xrightarrow{E_2} V_2 \) precisely coincide with morphisms \( T : V_1 \to V_2 \) of \( \text{Map}(C, D) \) with the property that all paths of the form \( E_1^*TE_2^* : V_1 \to V_2 \) compose to \( T \).