Definition (Multilinear Map). A multilinear map from a list of commutative monoids \(\langle A_1, 0_1, +_1 \rangle, \ldots, \langle A_n, 0_n, +_n \rangle\) to a commutative monoid \(\langle B, 0, + \rangle\) is a \(n\)-ary function \(f : A_1 \times \cdots \times A_n \to B\) satisfying the following properties for any index \(i \in \{1, \ldots, n\}\) and elements \(a_i \in A_i\) for\(i = 1, \ldots, n\) and \(a_{i-1} \in A_{i-1}\) and \(a_{i+1} \in A_{i+1}\):  
\[f(a_1, \ldots, 0_i, \ldots, a_n) = 0 \quad \text{and} \quad \forall a_i, a_i' \in A_i. f(a_1, \ldots, a_i + a_i', \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) + f(a_i, a_i', \ldots, a_n)\]

That is, a multilinear map is an \(n\)-ary function such that, for every index \(i\), fixing all the inputs besides the input for \(i\) results in a monoid homomorphism from \(\langle A_i, 0_i, +_i \rangle\) to \(\langle B, 0, + \rangle\). In particular, a unary multilinear map is simply a monoid homomorphism, and a nullary multilinear is simply a nullary function (i.e. an element of the codomain).

Exercise 1. Prove that \(\text{CommMon}\), comprised of commutative monoids and multilinear maps with identities and composition inherited from \(\text{Set}\), is a multicategory. In particular, demonstrate why the monoids must be commutative.

Exercise 2. Show that the internal monoids of \(\text{CommMon}\) bijectively correspond with semirings (defined in the previous homework).

Definition (Unit). An object \(I\) with a multimorphism \(\text{unit} : [\ ] \to I\) is called a unit object with a unit multimorphism if they form a tensor of the empty list. A multicategory has a unit if it has such an object and multimorphism.

Exercise 3. Prove that \(\langle \mathbb{N}, 0, + \rangle\) is a unit object of \(\text{CommMon}\). However, rather than showing the existence and uniqueness of \(\text{split}_{\vec{A}, \vec{B}} f\) for arbitrary lists of commutative monoids \(\vec{A}\) and \(\vec{B}\), for the sake of readability show this only for the case where \(\vec{A}\) is a singleton list and \(\vec{B}\) is empty.

Exercise 4. Prove that, for any given commutative monoids \(\langle A, 1, \cdot \rangle\) and \(\langle B, 1, \cdot \rangle\), the set of monoid homomorphisms from \(\langle A, 1, \cdot \rangle\) to \(\langle B, 1, \cdot \rangle\) is the underlying set of the left-exponential object \(\langle A, 1, \cdot \rangle \rightharpoonup \langle B, 1, \cdot \rangle\) in \(\text{CommMon}\). However, for the existence and uniqueness proof of \(\lambda f\), for the sake of readability only show this for the case where \(f\) is binary.

Exercise 5. Prove that, in \(\text{CommMon}\), the underlying set of the tensor of \(\langle A, 0, + \rangle\) and \(\langle B, 0, + \rangle\) is the set \(\mathcal{H}(A \times B)/\approx\), where \(\approx\) is the least equivalence relation satisfying:
\[
[0, b] \approx [\ ] \quad [(a + a', b)] \approx [(a, b), (a', b)] \quad [(a, 0)] \approx [\ ] \quad [(a, b + b')] \approx [(a, b), (a, b')] \\
\ell_1 \approx \ell_1' \land \ell_2 \approx \ell_2' \implies \ell_1 + \ell_2 \approx \ell_1' + \ell_2' \quad (\ell + \ell' \approx \ell' + \ell)
\]

Exercise 6. Prove that a unit of a cartesian multicategory is also a terminal object.