Multicategories

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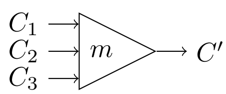
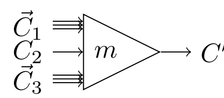
Definition (Multicategory)**.** A multicategory is comprised of the following components:

Objects A set Ob of "objects"

Multimorphisms: For every list of objects $\vec{C} \in \mathbb{L}\mathrm{Ob}$ and object $C' \in \mathrm{Ob}$, a set $\mathrm{Hom}(\vec{C}, C')$ of "multimorphisms from $\vec{C}$ to C'". Multimorphisms are often illustrated as shown below:



A multimorphism $m : [C_1, C_2, C_3] \to C'$



A multimorphism $m : \vec{C}_1 + [C_2] + \vec{C}_3 \to C'$

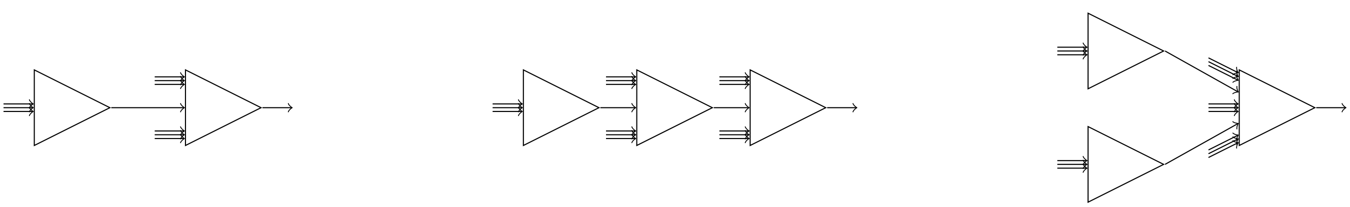
Identities For all objects $C \in \mathrm{Ob}$, a multimorphism $id_C : [C] \to C$

Composition For all (possibly empty) lists of multimorphisms $m_1 : \vec{C}_1 \to C'_1, \ldots, m_n : \vec{C}_n \to C'_n$ and multimorphisms $m' : [C'_1, \ldots, C'_n] \to C''$, a multimorphism $\triangle_{m_1,\ldots,m_n} m : \vec{C}_1 + \ldots + \vec{C}_n \to C''$

Identity For all multimorphisms $m : [C_1, \ldots, C_n] \to C'$, the equalities $\triangle_{id_{C_1},\ldots,id_{C_n}} m = m = \triangle_m id_{C'}$ hold

Associativity For all lists of lists of multimorphisms $\vec{m}_1, \ldots, \vec{m}_n$, lists of multimorphisms $m'_1, \ldots, m'_n$, and multimorphisms m'', the equality $\triangle_{\triangle_{\vec{m}_1} m'_1,\ldots,\triangle_{\vec{m}_n} m'_n} m'' = \triangle_{\vec{m}_1 + \ldots + \vec{m}_n} \triangle_{m'_1,\ldots,m'_n} m''$ holds when well typed

Remark. Just like the identity and associativity requirements of categories makes composing paths of morphisms unambiguous, the identity and associativity requirements of multicategories makes composing trees of multimorphisms unambiguous. Thus compositions of multimorphisms in multicategories are often described by illustration, as below:



Definition (Multigraph)**.** A multigraph is a set V of "vertices" and a set E of "edges" along with "source" and "target" functions $s : E \to \mathbb{L}V$ and $t : E \to V$. The category **Multigraph** is the comma category $\mathbf{Set}\downarrow((\mathbb{L}\cdot) \times \cdot)$.

Definition (Tree)**.** Given sets L and B, we inductively define the set $\mathbb{T}_L B$ as having two constructors $\mathsf{leaf}_B : L \to \mathbb{T}_L B$ and $\mathsf{branch}_L : \mathbb{L}(\mathbb{T}_L B) \times B \to \mathbb{T}_L B$.

Definition (Multipath)**.** Given a multigraph $\langle V, E, s, r\rangle$, we define functions $s_{\mathbb{T}} : \mathbb{T}_V E \to \mathbb{L}V$ and $t_{\mathbb{T}} : \mathbb{T}_V E \to V$ as follows:

$$s_{\mathbb{T}}(\mathsf{leaf}_E(v)) = [v] \qquad t_{\mathbb{T}}(\mathsf{leaf}_E(v)) = v$$
$$s_{\mathbb{T}}(\mathsf{branch}_V([t_1, \ldots, t_n], e)) = s_{\mathbb{T}}(t_1) + \ldots + s_{\mathbb{T}}(t_n) \qquad t_{\mathbb{T}}(\mathsf{branch}_V(\vec{t}, e)) = t(e)$$

A multipath of $\langle V, E, s, r\rangle$ is a tree $t \in \mathbb{T}_V E$ satisfying $\mathsf{colored}(t)$ as defined below:

$$\mathsf{colored}(\mathsf{leaf}_E(v)) = \mathbb{t}$$
$$\mathsf{colored}(\mathsf{branch}_V([t_1, \ldots, t_n], e)) = \mathsf{colored}(t_1) \wedge \cdots \wedge \mathsf{colored}(t_n) \wedge [t_{\mathbb{T}}(t_1), \ldots, t_{\mathbb{T}}(t_n)] = s(e)$$

This definition has an obvious extension to a monad $\mathsf{Multipath} : \mathbf{Multigraph} \to \mathbf{Multigraph}$.

Remark. Multicategories are the monad algebras of $\mathsf{Multipath}$.

Example. The multicategory **Set** has sets as its objects, and its multimorphisms from $[A_1,\ldots,A_n]$ to B are n-ary functions $f : A_1 \times \cdots \times A_n \to B$.

Example. **Prost** has preordered sets as its objects, and its multimorphisms from $[\langle A_1, \leq_1\rangle, \ldots, \langle A_n, \leq_n\rangle]$ to $\langle B, \leq\rangle$ are n-ary functions $f : A_1 \times \cdots \times A_n \to B$ satisfying the following:

$$\forall a_1, a_1' \in A_1, \ldots, a_n, a_n' \in A_n.\ a_1 \leq_1 a_1' \wedge \cdots \wedge a_n \leq_n a_n' \implies f(a_1,\ldots,a_n) \leq f(a_1',\ldots,a_n')$$

Example. **Rel**(2) has sets with binary relations as its objects, and its multimorphisms from $[\langle A_1, R_1\rangle, \ldots, \langle A_n, R_n\rangle]$ to $\langle B, S\rangle$ are n-ary functions $f : A_1 \times \cdots \times A_n \to B$ satisfying the following:

$$\forall a_1 \in A_1, \ldots, a_n \in A_n.\ \forall i \in \{1,\ldots,n\}, a_i' \in A_i.\ a_i\ R_i\ a_i' \implies f(a_1,\ldots,a_i,\ldots,a_n)\ S\ f(a_1,\ldots,a_i',\ldots,a_n)$$

Example. **Cat** has categories as its objects, and its multimorphisms from $[\mathbf{C}_1,\ldots,\mathbf{C}_n]$ to $\mathbf{D}$ are n-ary functors F, meaning F is a mapping of n-tuples of $\mathbf{C}$ objects to $\mathbf{D}$ objects and n-tuples of $\mathbf{C}$ morphisms to $\mathbf{D}$ morphisms (with obvious qualifications) such that:

$$F(id_{C_1},\ldots,id_{C_n}) = id_{F(C_1,\ldots,C_n)} \qquad F(f_1\,;g_1,\ldots,f_n\,;g_n) = F(f_1,\ldots,f_n)\,;F(g_1,\ldots,g_n)$$

Example. Given a grammar of types and a grammar of typed expressions with an appropriate notion of typed free variables and substitution, there is a corresponding multicategory whose objects are types τ, whose morphisms from $[\tau_1,\ldots,\tau_n]$ to τ' are expressions of type τ' with free variables of types $\tau_1,\ldots,\tau_n$, whose identities are given by variables, and whose composition is given by substitution.

Definition (Thin)**.** A multicategory is thin if all multimorphisms with the same domain and codomain are equal.

Definition (Multipreorder)**.** Just like a thin category corresponds to a preorder on its set of objects, a thin multicategory corresponds to a multipreorder on its set of objects. For example, the previous example could alternatively be specified by stating that $[m_1,\ldots,m_n] \leq m$ is defined as $m_1 * \cdots * m_n = m$. More explicitly, a multipreorder on a set A is a subset $\leq$ of $\mathbb{L}A \times A$ satisfying the following reflexivity and transitivity properties:

$$\forall a \in A.\ [a] \leq a$$

$$\forall \vec{a}_1,\ldots,\vec{a}_n \in \mathbb{L}A, a_1',\ldots,a_n', a'' \in A.\ \vec{a}_1 \leq a_1' \wedge \cdots \wedge \vec{a}_n \leq a_n' \wedge [a_1',\ldots,a_n'] \leq a'' \implies \vec{a}_1 \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} \vec{a}_n \leq a''$$

Note that an alternative definition for a multipreorder is a lax monad algebra of the monad $\mathbb{L}$ on **Rel**.

Example. Given a preordered monoid $\langle M, e, *, \leq\rangle$, there is a corresponding thin multicategory whose objects are the elements of M and which has a (unique) morphism from $[m_1,\ldots,m_n]$ to m if and only if $m_1 * \cdots * m_n \leq m$.

Definition (Multiorder)**.** A multiorder on a set A is a multipreorder on A with the following property:

$$\forall a, a' \in A.\ [a] \leq a' \implies a = a'$$

Definition (Operad)**.** An operad is a multicategory with exactly one object. As such, one often describes the domain of a multimorphism of an operad simply by its arity.

Example. The terminal multicategory **1** is the thin operad with exactly one morphism of each arity. In other words, it is the total multipreorder on the

Definition (Multifunctor)**.** A multifunctor F from a multicategory $\mathbf{C}$ to a multicategory $\mathbf{D}$ is a mapping of $\mathbf{C}$ objects to $\mathbf{D}$ objects and $\mathbf{C}$ multimorphisms to $\mathbf{D}$ multimorphisms (with obvious qualifications) such that:

$$F(id_C) = id_{F(C)} \qquad F\left(\underset{f_1,\ldots,f_n}{\triangle}\ g\right) = \underset{F(f_1),\ldots,F(f_n)}{\triangle}\ F(g)$$

Just as multicategories can equivalently be defined as monad algebras of $\mathsf{Multipath}$, multifunctors can equivalently be defined as morphisms of monad algebras of $\mathsf{Multipath}$.

Example. **Multicat** has multicategories as its objects, and its multimorphisms from $[\mathbf{C}_1,\ldots,\mathbf{C}_n]$ to $\mathbf{D}$ are n-ary multifunctors F, meaning F is a mapping of n-tuples of $\mathbf{C}$ objects to $\mathbf{D}$ objects and n-tuples of $\mathbf{C}$ multimorphisms to $\mathbf{D}$ multimorphisms (with obvious qualifications) such that:

$$F(id_{C_1},\ldots,id_{C_n}) = id_{F(C_1,\ldots,C_n)} \qquad F\left(\underset{f_1^1,\ldots,f_1^m}{\triangle}\ g_1,\ldots,\underset{f_n^1,\ldots,f_n^m}{\triangle}\ g_n\right) = \underset{F(f_1^1,\ldots,f_n^1),\ldots,F(f_1^m,\ldots,f_n^m)}{\triangle}\ F(g_1,\ldots,g_n)$$

Definition (Internal Monoid of a Multicategory)**.** An internal monoid of a multicategory **C** is a multifunctor from the terminal multicategory **1** to **C**. That is, an internal monoid of **C** is an object C of **C** along with a collection of multimorphims $\{*_n : [C, \ldots, (i \text{ occurences of } C)] \to C\}_{n \in \mathbb{N}}$ such that all compositions of the multimorphisms in $\{*_n\}_{n \in \mathbb{N}}$ with the same codomain are equal. One can prove that an internal monoid could equivalently by defined as an object C with two multimorphisms $*_0 : [\,] \to C$ and $*_2 : [C, C] \to C$ satisfying $\Delta_{id_C, *_0} *_2 = id_C = \Delta_{*_0, id_C} *_2$ and $\Delta_{*_2, id_C} *_2 = \Delta_{id_C, *_2} *_2$.

Example. An internal monoid of **Set** is simply a monoid. An internal monoid of **Prost** is a preordered monoid, meaning $\forall a_1, a_1', a_2, a_2' \in A.\ a_1 \leq a_1' \wedge a_2 \leq a_2' \implies a_1 * a_2 \leq a_1' * a_2'$.

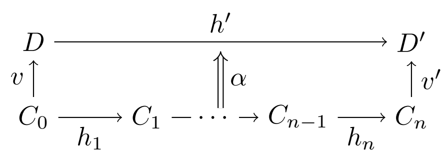
Definition (Path-Multicategory)**.** There is a common pattern of multicategories whose objects have left and right "anchors". Furthermore, often wants consider multifunctors that preserve these anchors. As such, its best to formalize this pattern as a Path-multicategory, more commonly known as an fc-multicategory (where fc stands for "free category") or a virtual double category. A Path-multicategory is comprised of the following components:

0-cells A set Ob of "0-cells" or "objects"

Vertical 1-cells For every pair of 0-cells C and D, a set $\mathrm{Vom}(C, D)$ of "vertical 1-cells from C to D"

Horizontal 1-cells For every pair of 0-cells C and C', a set $\mathrm{Hom}(C, C')$ of "horizontal 1-cells from C to C'"

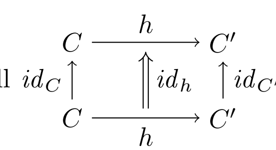
2-cells For every path of horizontal 1-cells $C_0 \xrightarrow{h_1} C_1 \ldots C_{n-1} \xrightarrow{h_n} C_n$, every horizontal 1-cell $D \xrightarrow{h'} D'$, and every vertical 1-cells $v : C_0 \to D$ and $v' : C_n \to D'$, a set $\mathrm{Face}_{v,v'}([h_0, \ldots, h_n], h')$ of "2-cells from $[h_1, \ldots, h_n]$ to h' along v and v'". 2-cells are often illustrated as shown below:



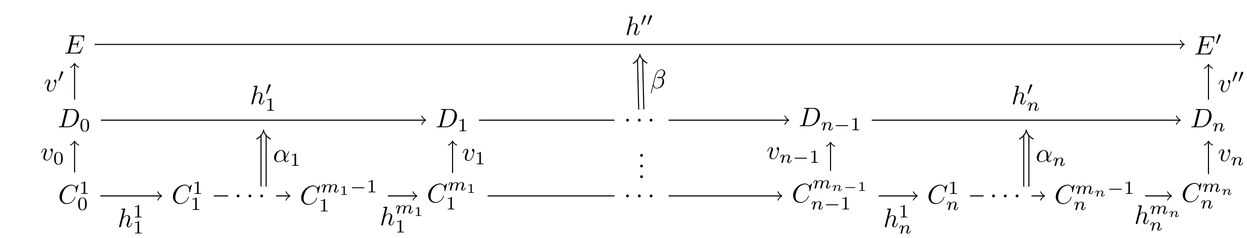
Vertical 1-identities For every 0-cell C, a vertical 1-cell $id_C : C \to C$

Vertical 1-composition For every sequence of vertical 1-cells $C \xrightarrow{v} D \xrightarrow{v'} E$, a vertical 1-cell $C \xrightarrow{v\,;\,v'} E$

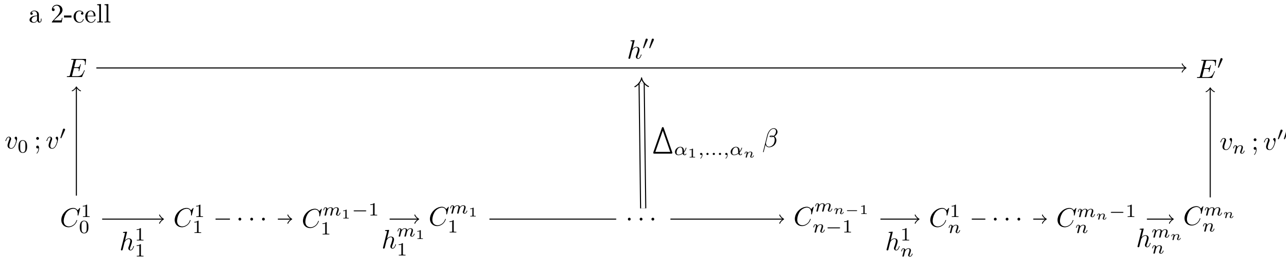
2-identities For every horizontal 1-cell $h : C \to C'$, a 2-cell



2-composition For every "tree" of 2-cells



a 2-cell



1-Identity $\forall v : C \to D.\ id_C \,;\, v = v = v \,;\, id_D$

1-Associativity $\forall C \xrightarrow{v} D \xrightarrow{v'} E \xrightarrow{v''} F.\ (v\,;\,v')\,;\,v'' = v\,;(v'\,;\,v'')$

2-Identity $\forall \alpha : [h_1, \ldots, h_n] \Rightarrow h'.\ \Delta_{id_{h_1}, \ldots, id_{h_n}} \alpha = \alpha = \Delta_\alpha\, id_{h'}$

2-Associativity $\forall \vec{\alpha}_1, \ldots, \vec{\alpha}_n, \beta_1, \ldots, \beta_n, \gamma.\ \Delta_{\Delta_{\vec{\alpha}_1} \beta_1, \ldots, \Delta_{\vec{\alpha}_n} \beta_n} \gamma = \Delta_{\vec{\alpha}_1 + \ldots + \vec{\alpha}_n} \Delta_{\beta_1, \ldots, \beta_n} \gamma$

Example. **BinRel** has sets as its objects, functions as its vertical morphisms, and binary relations as its horizontal morphisms. It has a unique 2-cell from $[R_1 \subseteq A_0 \times A_1, \ldots, R_n \subseteq A_{n-1} \times A_n]$ to $S \subseteq B \times B'$ along $f : A_0 \to B$ and $f' : A_n \to B'$ if and only if the following holds:

$$\forall a_0 : A_0, \ldots, a_n : A_n.\ a_0\ R_1\ a_1 \wedge \cdots \wedge a_{n-1}\ R_n\ a_n \implies f(a_0)\ S\ f'(a_n)$$

Example. **SplitGraph** has sets "of vertices" as its objects and "vertex mapping" functions as its vertical morphisms. A horizontal morphisms from V to V' is a set E of "edges" along with a "source" function $s : E \to V$ and a "target" function $t : E \to V'$. Given a chain of horizontal morphisms $[V_0 \xleftarrow{s_1} E_1 \xrightarrow{t_1} V_1, \ldots, V_{n-1} \xleftarrow{s_n} E_n \xrightarrow{t_n} V_n]$, a "path" of this chain is a sequence $[v_0 \in V_0, e_1 \in E_1, v_1 \in V_1, \ldots, v_{n-1} \in V_{n-1}, e_n \in E_n, v_n \in V_n]$ with the property that $\forall i \in \{1, \ldots, n\}.\ v_{i-1} = s_i(e_i) \wedge t_i(e_i) = v_n$. A 2-cell from $[V_0 \xleftarrow{s_1} E_1 \xrightarrow{t_1} V_1, \ldots, V_{n-1} \xleftarrow{s_n} E_n \xrightarrow{t_n} V_n]$ to $V' \xleftarrow{s'} E' \xrightarrow{t'} V''$ along $f_V : V_0 \to V'$ and $f'_V : V_n \to V''$ is a "path mapping" function f_E mapping each path $[v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n]$ of the domain to an element e' of E' such that $f_V(v_0) = s'(e')$ and $t'(e') = f'_V(v_n)$.

Example. Given a 2-category, there is a corresponding Path-multicategory with the same 0-cells, whose horizontal and vertical 1-cells are both the 1-cells of the 2-category, and whose 2-cells from $[h_1, \ldots, h_n]$ to h' along v and v' are the 2-cells of the 2-category from $h_1 ; \ldots ; h_n ; v'$ to $v ; h'$.

Example. The terminal Path-multicategory **1** has one object, one vertical morphism (which is the identity on that object), one horizontal morphism, and one 2-cell for each path of horizontal morphisms (i.e. one 2-cell of arity n for every natural number n).

Definition (Path-Multifunctor)**.** A Path-multifunctor F from a Path-multicategory $\mathbf{C}$ to a Path-multicategory $\mathbf{D}$ is a mapping of $\mathbf{C}$ objects to $\mathbf{D}$ objects, $\mathbf{C}$ vertical multimorphisms to $\mathbf{D}$ vertical morphisms, $\mathbf{C}$ horizontal morphisms to $\mathbf{D}$ horizontal morphisms, and $\mathbf{C}$ 2-cells to $\mathbf{D}$ 2-cells (with obvious qualifications) such that:

$$F(id_C) = id_{F(C)} \qquad F(v ; v') = F(v) ; F(v') \qquad F(id_h) = id_{F(h)} \qquad F\left(\mathop{\triangle}_{\alpha_1,\ldots,\alpha_n} \beta\right) = \mathop{\triangle}_{F(\alpha_1),\ldots,F(\alpha_n)} F(\beta)$$

Definition (Internal Monoid of a Path-Multicategory)**.** An internal monoid of a Path-multicategory $\mathbf{C}$ is a Path-multifunctor from the terminal multicategory **1** to $\mathbf{C}$. That is, an internal monoid of $\mathbf{C}$ is an object C of $\mathbf{C}$ along with a horizontal morphism $m : C \to C$ and a collection of 2-cells $\{\mu_n \in \text{Face}_{id_C, id_C}([m, \ldots, (i \text{ occurences of } m)], m)\}_{n \in \mathbb{N}}$ such that all compositions of the 2-cells in $\{\mu_n\}_{n \in \mathbb{N}}$ with the same codomain are equal. One can prove that an internal monoid could equivalently by defined as an object C with a horizontal morphism $m : C \to C$ and two 2-cells $\mu_0 \in \text{Face}_{id_C, id_C}([\,], m)$ and $\mu_2 \in \text{Face}_{id_C, id_C}([m, m], m)$ satisfying $\triangle_{id_m, \mu_0} \mu_2 = id_m = \triangle_{\mu_0, id_m} \mu_2$ and $\triangle_{\mu_2, id_m} \mu_2 = \triangle_{id_m, \mu_2} \mu_2$.

Example. An internal monoid of **BinRel** is a preordered set. An internal monoid of **SplitGraph** is a category. An internal monoid of a 2-category as a Path-multicategory is a monad in that 2-category.

Definition (Natural Transformation of Multifunctors)**.** Given multifunctors F and G from multicategory $\mathbf{C}$ to multicategory $\mathbf{D}$, a natural transformation from F to G is a collection of multimorphisms $\{\alpha_C : [F(C)] \to G(C)\}_{C \in \mathbf{C}}$ satisfying the property $\forall f : [C_1, \ldots, C_n] \to C'.\ \triangle_{\alpha_{C_1}, \ldots, \alpha_{C_n}} G(f) = \triangle_{F(f)} \alpha_{C'}$.

Definition (Internal Monoid Homomorphism of a Multicategory)**.** Given two internal monoids of a multicategory $M, M' : \mathbf{1} \to \mathbf{C}$, an internal monoid homomorphism is a natural transformation from M to M'.

Definition (Natural Transformation of Path-Multifunctors)**.** Given Path-multifunctors F and G from $\mathbf{C}$ to $\mathbf{D}$, a natural transformation from F to G consists of a collection of vertical morphisms $\{\nu_C : F(C) \to G(C)\}_{C \in \mathbf{C}}$ and a collection of 2-cells $\{\theta_h \in \text{Face}_{\nu_C, \nu_{C'}}([F(h)], G(h))\}_{h \in \text{Hom}_{\mathbf{C}}(C, C')}$ satisfying the following properties:

$$\forall v : C \to C'.\ \nu_C ; G(v) = F(\nu) ; \alpha_{C'} \qquad \text{and} \qquad \forall \alpha : [h_1, \ldots, h_n] \Rightarrow h'.\ \mathop{\triangle}_{\theta_{h_1}, \ldots, \theta_{h_n}} G(\alpha) = \mathop{\triangle}_{F(\alpha)} \theta_{h'}$$

Definition (Modification of Natural Transformations of Path-Multifunctors)**.** Given natural transformations $\langle \nu, \theta \rangle$ and $\langle \nu', \theta' \rangle$ from Path-multifunctor F to Path-multifunctor G from $\mathbf{C}$ to $\mathbf{D}$, a modification from $\langle \nu, \theta \rangle$ and $\langle \nu', \theta' \rangle$ is a collection of 2-cells $\{\mu_h \in \text{Face}_{\nu_C, \nu'_{C'}}([F(h)], G(h))\}_{h \in \text{Hom}_{\mathbf{C}}(C, C')}$ satisfying the following property:

$$\forall \alpha : [C_0 \xrightarrow{h_1} C_1 \ldots C_{i-1} \xrightarrow{h_i} C_i \ldots C_{n-1} \xrightarrow{h_n} C_n] \Rightarrow h'.\ \mathop{\triangle}_{\theta_{h_1}, \ldots, \mu_{h_i}, \ldots, \theta_{h_n}} G(\alpha) = \mathop{\triangle}_{F(\alpha)} \mu_{h'}$$

Example. Just as monads correspond to Path-multifunctors from **1** to a 2-category, oplax monad morphisms between monads correspond to natural transformations between those Path-multifunctors.

Example. Just as preorders correspond to Path-multifunctors from **1** to **BinRel**, relation-preserving function between preorders correspond to natural transformations between those Path-multifunctors, and a relation-preserving function is less than another in the usual sense if and only if there exists a (necessarily unique) modification between the corresponding natural transformations.

Example. Just as categories correspond to Path-multifunctors from **1** to **SplitGraph**, functors between categories correspond to natural transformations between those Path-multifunctors, and natural transformations between functors correspond to modifications between those natural transformations.

Remark. Given a multicategory, there is a corresponding Path-multicategory called its *suspension* with exactly one 0-cell, whose only vertical morphism is the identity on that unique 0-cell, whose horizontal morphisms (necesarrily from that unique 0-cell to itself) are the objects of the multicategory, and whose 2-cells are the multimorphisms of the multicategory.

In the other direction, given a Path-multicategory there is a corresponding multicategory whose objects are the horizontal 1-cells and whose multimorphisms are the 2-cells, a construction that in a sense forgets the 0-cells and the vertical 1-cells. Suspension and this forgetful construction can be extended to mappings between (Path-)multifunctors, between transformations, and even between modifications, provided one adapts the above definition of modifications for Path-multicategories to plain multicategories in the obvious manner. Putting aside modifications for simplicity, both of these mappings preserve the obvious forms of composition on these structures, meaning they are both 2-functors between the respective 2-categories **Multicat** and Path-**Multicat**. In fact, the suspension 2-functor is right adjoint to the forgetful 2-functor (in the 2-category of 2-categories).